Process Fluid Flow 2022-2023

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Contents

List of Figures

List of Tables

Nomenclature

Roman

A	Area	m^2
A_c	General cross-sectional area	m^2
b	Open-channel base width	m
C_D	Discharge coefficient	–
D	Diameter	m
d	Open-channel fluid depth	m
D_H	Hydraulic diameter	m
E	Energy	J
E	Pipe bulk modulus of elasticity	Pa
$\mathbf{f}$	External force per unit mass	$N\,kg^{-1}$
F	Force	N
f	Fanning friction factor	–
g	Acceleration due to gravity	$9.81\,m\,s^{-2}$
h	Height	m
h_o	Height of the base of an open-channel	m
$\mathbf{i}$	Unit vector in x-direction	–
$\mathbf{j}$	Unit vector in y-direction	–
$\mathbf{k}$	Unit vector in z-direction	–
K	Fluid bulk modulus of elasticity	Pa
K	Number of velocity heads	m
L	Length	m
l	Displacement	m
l	Distance	m
m	Mass	kg
$\dot{M}$	Mass flow rate	$kg\,s^{-1}$
$\mathbf{n}$	Outward-pointing normal vector from a surface	–
n	Moles	mol
P	Pressure	Pa
P	Wetted perimeter length	m
p	Vapour pressure	Pa
Q	Volumetric flow rate	$m^3\,s^{-1}$
$\dot{Q}$	Heat loss	W
R	Gas constant	$8.314\,J\,mol^{-1}\,K^{-1}$
R	Radius	m
r	Radial position	m
R_b	Bend radius	m

R_H	Hydraulic radius	m
R^*	Hydraulic resistance	$s^2\,m^{-5}$
S	Streamwise slope	$m\,m^{-1}$
s	General source or sink	depends
S_f	Friction slope	$m\,m^{-1}$
T	Temperature	K
T	Width of the free surface of the flow	m
t	Time	s
t	Time	s
t_w	Wall thickness	m
u	Velocity in x-direction	$m\,s^{-1}$
$\mathbf{V}$	Velocity field	$m\,s^{-1}$
V	Volume	m^3
v	Velocity in the y-direction	$m\,s^{-1}$
W	Work	J
W	Work	W
w	Velocity in the z-direction	$m\,s^{-1}$
x	x-direction	m
x	General co-ordinate system	–
y	y-direction	m
y	General co-ordinate system	–
z	z-direction	m
z	General co-ordinate system	–
Greek		
α	Open-channel slope angle	rad
α	angle	–
β	angle	–
Δh_f	Head loss due to friction	m
Δh_p	Pump head	m
ε	Surface roughness	m
Γ	General boundary	m^2
γ	Shear Strain	–
$\dot{\gamma}$	Shear rate	s^{-1}
μ	Viscosity	Pa s
Ω	General volume	m^3
φ	General property	depends
ρ	Density	$kg\,m^{-3}$
σ	Normal stress	Pa
σ	Surface tension	$N\,m^{-1}$
$\boldsymbol{\sigma}$	Stresses on a element	Pa
τ	Shear stress	Pa
τ_w	Wall shear stress	Pa
θ	Angle	–
Dimentionless Variables		
Fr	Froude number	$u/(gA_c/T)^{0.5}$
Re	Reynolds number	$Du\rho/\mu$

Course Introduction

Chemical engineers are frequently interested in systems involving the flow of fluids. This interest occurs at two levels:

1. the practical level, in which many of the methodologies for the design of chemical engineering processes and operations require fluid flow calculations; and
2. the conceptual level, in which fluid flow illustrates one of the distinctive and defining skills of a chemical engineer, that of being able to take a fundamental understanding of the physical universe and apply it to solve practical problems in a way that is both effective and elegant.

Practical examples of fluid flow calculations include pipe network design (including sizing of pumps and pipelines for transferring fluids within processes) and fluid flow effects on rates of heat and mass transfer, on reaction rates and on separation systems.

Conceptually, fluid flow is an example of a transport process, in which the rate of transfer of matter or energy depends on physical factors affecting the transfer, such as the physical properties of the material under investigation and the geometry of the system.

Some of the key fundamental ideas needed for fluid flow are the Conservation Laws, of which the three most important ones are:

Conservation of Mass

Conservation of Energy

Conservation of Linear Momentum

These three conservation laws will form the basis for developing our fundamental understanding of Fluid Flow, as later on we will employ these fundamental ideas in order to derive the three major mathematical descriptions of Fluid Flow: the Continuity Equation, Bernoulli's Equation, and the Momentum Equation. We will derive these firstly for the simplified case of Ideal Fluids, then extend the equations to allow us to do useful calculations involving Real Fluids. We will illustrate the use of fluid flow in the context of chemical engineering using examples relating to flow in pipes and vessels. Before we move to flowing fluids, we need to have some understanding of the static behaviour of fluids, which is the study of Hydrostatics; and before we begin with that, we need to formally identify what fluids are and be introduced to some of their important properties.

Some of the mathematical proofs/derivations in this course may look very intimidating at first glance. However, don't let this put you off; in this course we are not aiming to solve complex vector calculus, we are simply using the correct mathematical notation which we will then simplify into key equations needed to solve all the problems. Make sure you understand what the terms represent/mean rather than worrying about the mathematics.

During this course we will also try to provide problems based around actual tasks/designs that professional engineers may need to undertake.

Chapter 1

Introduction to Fluids

Contents

1.1 Chapter 1 ILOs

ILO 1.1. Recall the definition of a fluid and that they can be analysed with fluid mechanics.

ILO 1.2. Describe basic properties of fluids.

1.2 Fluids

1.2.1 What is a Fluid?

Generally, matter exists in mainly one of four fundamental states (though there are many more): solid, liquid, gas, or plasma (although materials with intermediate properties exist, in particular, materials that exhibit both solid-like and fluid-like properties e.g. liquid crystals), Figure 1.1 [1]. A fluid is a liquid or a gas, which differ from solids in that,

> **Fluids**
>
> Fluids deform continuously under the action of an applied shear stress.

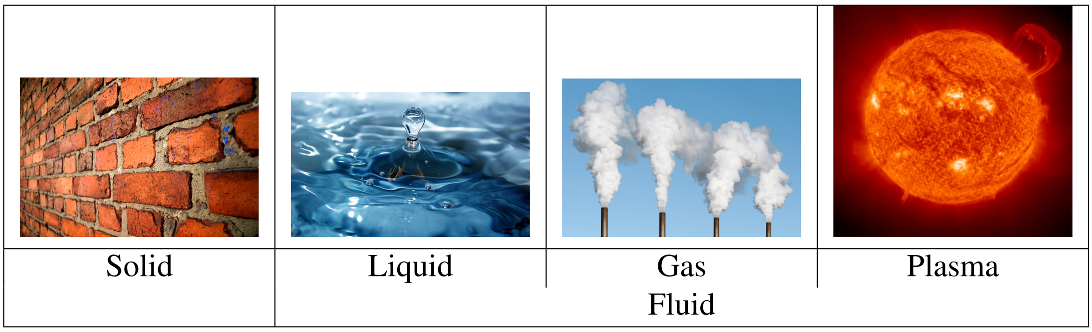


Figure 1.1: The four common states of matter.

A shear stress is a force that acts on an area tangential to (as opposed to perpendicular to) the direction of the force (this will be explored more in Chapter 3). If a shear force is applied to a solid, it may deform a little from its initial position, but it will not continue to do so, and when the force is removed, it will recover its initial shape. By contrast, a fluid will continue to deform as long as the shearing action is applied, and when the shear force is removed, it will not bounce back to its original position. (A viscoelastic material is in between – it will flow to some extent on application of a shear force, and will bounce back to some extent, but not completely, on removal of the applied force - will will not cover these types of materials during this course. Blu-tack and bread doughs are both examples of viscoelastic materials.)

Fluids may be subdivided into liquids and gases (and plasma), which differ from each other in several distinctive ways. The molecules in liquids stick closely together such that liquids have a definite volume and will not fill a container having a larger volume. The volume occupied by a liquid varies only slightly with temperature and pressure, such that liquids are essentially incompressible. Liquids form interfaces with gases, including their own vapours, and sometimes with other liquids (e.g. oil-water mixtures). By contrast, the molecules of gases will always move apart so as to completely fill an enclosing container; only then is a gas in equilibrium.

With respect to their fluid behaviour, gases differ from liquids in two significant respects. Firstly, while liquids are essentially incompressible for most practical purposes, gases by contrast are easily compressible. Secondly, liquids have much greater densities than gases, such that the weight of a liquid plays a major role in its behaviour, while the effects of weight can usually be ignored when dealing with gases.

1.2.2 Fluid Mechanics

Fluid mechanics is the application of the fundamental principles of mechanics and thermodynamics – such as conservation of mass, conservation of energy and Newton's laws of motion – to the study of liquids and gases, in order to explain observed phenomena and to be able to predict behaviour. Fluid mechanics can be sub-divided into Fluid Statics (or Hydrostatics) – the study of fluids at rest – and Fluid Dynamics (or Hydrodynamics), the study of fluids in motion. This course is entitled Process Fluid Flow, emphasising the issues of fluid behaviour under dynamic conditions, because chemical engineering is concerned principally with processes, and processes imply change and are inherently dynamic. We need to understand fluid statics (or hydrostatics) as well.

1.2.3 Fluid Flow in Chemical Engineering Applications

Figure 1.2 illustrates a simple chemical process incorporating a feed stream into a reactor, in which an exothermic reaction results in formation of a desired product and an unwanted byproduct, and causes an increase in the temperature of the contents of the reactor. The stream from the reactor is cooled and then subject to an initial separation operation in order to recover the unreacted feed, which is recycled back to the reactor in order to increase the conversion of feed to product. A second separation system then separates the desired product from the undesired byproduct.

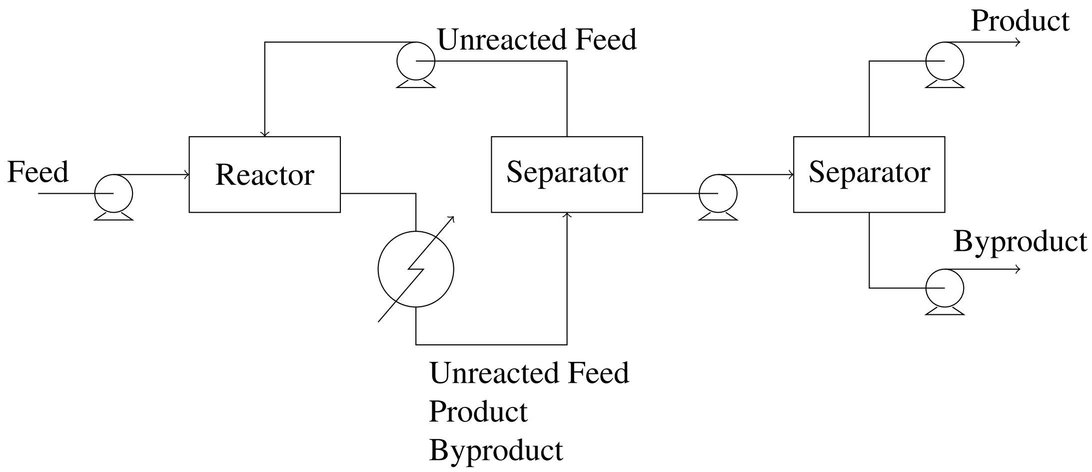


Figure 1.2: Example of a simple chemical process.

This example allows us to identify the relevance of fluid flow to numerous aspects of chemical process design:

Pipes – streams are conveyed between processes via numerous pipes, which contribute a large fraction of the capital costs of a chemical plant, and so need to be sized appropriately.

Pumps – make the fluids flow along the pipes! They need to be sized to deliver the required flowrates, while their power consumption contributes to the operating costs of the plant.

Reactor – the nature of the fluid flow patterns determines the way that molecules meet each other and react, and hence the yield, throughput and selectivity of the process.

Heat exchangers – only one shown here (a cooling operation), but there are many in a typical process, and the rate of heat transfer, and therefore the size and cost of the heat exchanger, depends strongly on the fluid flow patterns.

Separation – probably via distillation, in which flow patterns of the liquid and vapour phases affect mass transfer between them.

Fluid flow also has an enormously wide range of application outside of chemical engineering, including in the other engineering disciplines (in particular, civil and mechanical engineering), in meteorology and oceanography (weather patterns and ocean currents), and even in medicine (e.g. blood flow). Mastery of fluid flow can open up a wide range of employment and research opportunities both within the chemical industries and beyond.

1.3 Properties of Fluids

Density (ρ) **–** mass per unit volume,

$$\rho = \frac{m}{V} \tag{1.3.1}$$

In SI units, density has the units of $\text{kg}\,\text{m}^{-3}$, because it is compressible, the density of a gas depends on its pressure. Relative density or specific gravity is the ratio of the density of a material to the density of water at 4°C. Because it is a ratio, it does not have units.

Pressure (P) **–** Fluids have pressure, and fluids flow because of pressure differences. An understanding of what pressure is and how it relates to fluids is key to understanding the subject of fluid flow, this will be covered more in Chapter 2. A key equation to help explain pressure is,

$$\Delta P = \rho g \Delta h \tag{1.3.2}$$

where ΔP is a pressure difference, ρ is the density we have already met, g is the acceleration due to gravity (9.81 $\text{m}\,\text{s}^{-2}$ in SI units), and Δh is the height over which the pressure difference is being measured. By considering the units, we can see that the units of pressure are:

$$\frac{\text{kg}}{\text{m}^3} \times \frac{\text{m}}{\text{s}^2} \times \text{m} = \frac{\text{kg}}{\text{m}\,\text{s}^2}$$

Pressure is often presented in other units as well.

Vapour Pressure (p) **–** Liquids exhibit a vapour pressure, which contributes to the total pressure above the liquid. Vapour pressures are not considered within this course.

Viscosity (μ) **–** This is the fluid property responsible for resistance to applied forces. Honey has high viscosity, water has much lower viscosity, and air has an even lower viscosity. Viscosity will be covered in more detail in Chapter 3, but for now we'll note that its SI units are $\text{kg}\,\text{m}^{-1}\,\text{s}^{-1}$ or Pa s.

Surface Tension (σ) **–** Liquids in contact with gases (e.g. water in contact with air) form an interface. The liquid molecules at the interface are attracted to each other more than to the gas molecules, to they tend to pull sideways. This has the effect of minimising the size of the interface. For this reason, bubbles are spherical, because surface tension aims to minimise the interface between the liquid that makes up the bubble wall and the air inside and outside of the bubble, and a sphere is the shape that has the smallest surface area for a given volume. However, surface tension effects are outside the scope of this course.

1.4 References

[1] Holland, F. and Bragg, R. [1995], *Fluid Flow for Chemical Engineers*, Butterworth-Heinemann.

Chapter 2

Hydrostatics

Contents

2.1 Chapter 2 ILOs

ILO 2.1. Define pressure.

ILO 2.2. Calculate hydrostatic pressure in terms of fluid densities.

ILO 2.3. Compare different methods of pressure measurement.

2.2 Introduction

Hydrostatics is the study of fluids at rest, for which there are no shear (tangential) forces. However, fluids at rest are still subject to pressure, as pressure arises as the result of innumerable molecular collisions. Pressure is therefore where our study of hydrostatics begins. Then, as we will discover, hydrostatics is also the basis for measurement of pressure.

2.3 Pressure

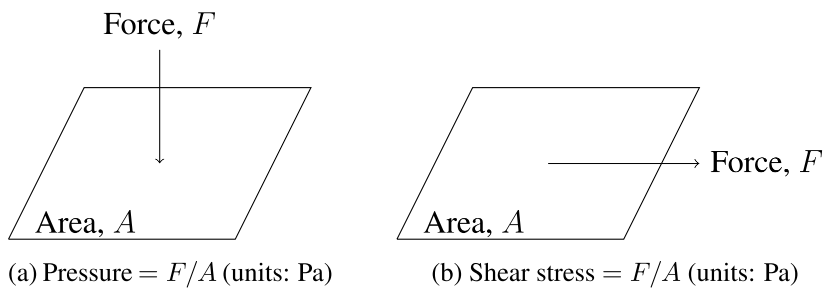
Understanding pressure and how it relates to fluids is key to understanding the subject of fluid flow. Pressure is defined as the force per unit area,

Pressure

$$P = \frac{F}{A} \tag{2.3.1}$$

Pressure therefore has SI units of Newtons per metre squared, $\mathrm{N\,m^{-2}}$, which we simplify to Pascals, Pa; 1 Pa = $1\,\mathrm{N\,m^{-2}}$. This is a very small value, approximately equivalent to the force exerted by a 2 g weight spread over the surface of your hand. The unit kPa (10^3 Pa) is therefore more often used. Standard atmospheric pressure has a value of 101.325 kPa.

Interestingly, shear stress is also a force per unit area, and also has units of Pa. The difference between these is the direction of the force (we will see more of shear stress in section 3.4. In the case of pressure, the direction of the force is perpendicular to (or normal to) the area, while in the case of shear stress, the direction of the force is tangential to the area.



(a) Pressure $= F/A$ (units: Pa) (b) Shear stress $= F/A$ (units: Pa)

Figure 2.1: Pressure and shear stress.

Pressure is a scalar quantity (has magnitude but no direction), while Force is a vector (has both magnitude and direction). So when the Pressure acts on an Area, this has an orientation (i.e. a direction), which determines the direction of the resulting Force.

We sometimes say, slightly imprecisely, that the pressure at a particular point in a fluid at rest is the same in all directions. But because pressure is a scalar quantity, it is meaningless to talk of the pressure having a particular direction. It is more precise to say that the magnitude of the force due to the pressure is independent of the orientation of the plane over which the force acts. This is in effect the same as saying that the pressure at a point is the same in all directions (provided we understand that we actually mean that the magnitude of the force arising from the pressure at that point is the same in all directions). This is strictly true only for fluids at rest, but close enough to the truth even for fluids in motion. Another way of putting this idea is that the magnitude of the pressure is independent of the orientation of the surface used to define it.

Let's have another look at that definition of pressure, and focus on the units,

$$P = \frac{F}{A};\ \text{units}\ \frac{\text{N}}{\text{m}^2} \times \frac{\text{m}}{\text{m}} = \frac{\text{N m}}{\text{m}^3}$$

In this case we have multiplied the top and bottom of the fraction by meters, this shows Newtons times metres on the top, which is a Force times a distance, which equals Work, which is a form of Energy. So the top part has units of energy (which are Joules, J). The bottom part has units of Volume (m^3), so the whole indicates that the units of pressure, as well as indicating force per unit area, have an equivalent meaning of energy per unit volume (J m^{-3}). This is also apparent when we look at the units of the ideal gas law,

$$PV = nRT \tag{2.3.2}$$

The units of the universal gas constant, R, are J mol^{-1} K^{-1}. On the right hand side of the equation, this is multiplied by the number of moles (mol) and by the temperature (K) to give J, which is the unit of energy. The left hand side must also therefore multiply to give units of energy. V is volume, in m^3, so the pressure term, P, must represent in some sense energy per unit volume, with units of J m^{-3}:

$$\frac{\text{J}}{\text{m}^3} \times \text{m}^3 = \text{J} = \text{mol} \times \frac{\text{J}}{\text{mol K}} \times \text{K}$$

So pressure appears to be a form of energy. Now, we have to have a slight word of caution, that some textbooks insist pressure is not a type of energy that a fluid possesses, and that it is incorrect to speak of "pressure energy". However, while recognising that pressure does not precisely mean "Energy per unit Volume", at the same time it is quite helpful sometimes to think of pressure in this way, instead of our usual way of thinking of pressure as Force per unit Area, and to recognise that the units of pressure, N m^{-2}, are equivalent to J m^{-3}.

Two complications arise when dealing with the practical measurement and interpretation of pressure. The first is that, although the SI units of pressure are Pa (or kPa), many other units for measuring and reporting pressure are commonly in use. These include pounds per square inch (psi), Torr, mm Hg, atmosphere (atm), bar, m H_2O. Get used to seeing pressure expressed in these various alternative units, and get used to converting them to Pa using your conversion tables or appropriate equations.

Of these, the bar needs a special mention. It has a value of 10^5 Pa. This makes it an unusual measurement in the context of the SI system, which usually uses multiples and submultiples of the fundamental units which vary by powers of 3 (e.g. kilo, mega and giga, 10^3, 10^6 and 10^9, respectively, or milli, micro and nano, 10^{-3}, 10^{-6}, and 10^{-9}, respectively). However, the bar is convenient in that it is very close to atmospheric pressure (1.01325×10^5 Pa), such that 1 bar is approximately equal to 1 atmosphere. Hence the bar is commonly used in science and engineering (as is the millibar, which is 10^{-3} bar or 10^2 Pa).

2.3.1 Types of Pressure

The second complication regarding pressure is that it is commonly measured relative to one of two different datum points – and when we are dealing with pressure measurements, we need to be aware which datum point is being employed or assumed. The two datum points are,

- relative to atmospheric pressure; and
- relative to an absolute pressure of zero (absolute vacuum).

The first of these is convenient, as we are generally measuring the pressure of things that are situated on the ground and surrounded by the atmosphere, and we want to know whether they are under pressure or perhaps under vacuum, relative to the surrounding atmosphere. Things that are at the same pressure as the atmosphere we would say are not under pressure, and that their pressure is zero (meaning, zero relative to the surrounding atmospheric pressure). Things that are above atmospheric pressure are under positive pressure, and things that are below are under a partial vacuum or a negative pressure. Such a pressure measurement is called a “gauge pressure”.

However, in other circumstances, we want to know what the absolute value of the pressure is, not just the pressure relative to atmospheric pressure. Otherwise, for example, how would we know that the value of standard atmospheric pressure is 101.325 kPa? This value only has meaning relative to an absolute pressure of zero. Atmospheric pressure in fact changes, as we know, with geographical position and over time. This is why we need to define “standard” atmospheric pressure. But measuring a quantity relative to something that changes with time or is different in different places is not a very solid basis for science or engineering. Absolute vacuum, a pressure of zero, does not change, so measuring pressures relative to this constant datum point is much less ambiguous and a much sounder basis. Pressures measured and reported relative to absolute vacuum are called “absolute pressures”, in contrast to “gauge pressures”.

The pressure type may be made explicit in the reporting (“50 kPa gauge” or “20 kPa abs”, “1 barg” (bar gauge) or “2 bara” (bar absolute), “15 psig” or “30 psia”), or may be omitted, with the engineer expected to know from the context whether a gauge or absolute pressure is being used. Clearly, the numerical values of absolute pressures are larger than those of gauge pressures, by an amount equal to the absolute atmospheric pressure (in whichever units are being used),

Pressure Types

$$\begin{array}{ccccc} \text{Absolute pressure} & = & \text{Gauge pressure} & + & \text{Atmospheric pressure} \\ P_{\text{abs}} & = & P_{\text{gauge}} & + & P_{\text{atm}} \end{array} \tag{2.3.3}$$

So, both practices are used, and you need to be aware of both and of the relationship between them, and very aware of which is being used in the particular context of the calculation you are performing or the problem you are addressing. Note that many properties of gases are functions of the absolute pressure, and so in problems concerning gases, or in using the ideal gas equation $PV = nRT$, absolute pressures should be used.

Figure 2.2 illustrates the relationship between gauge and absolute pressures, and the equivalence or approximate equivalence of several common units for reporting pressure.

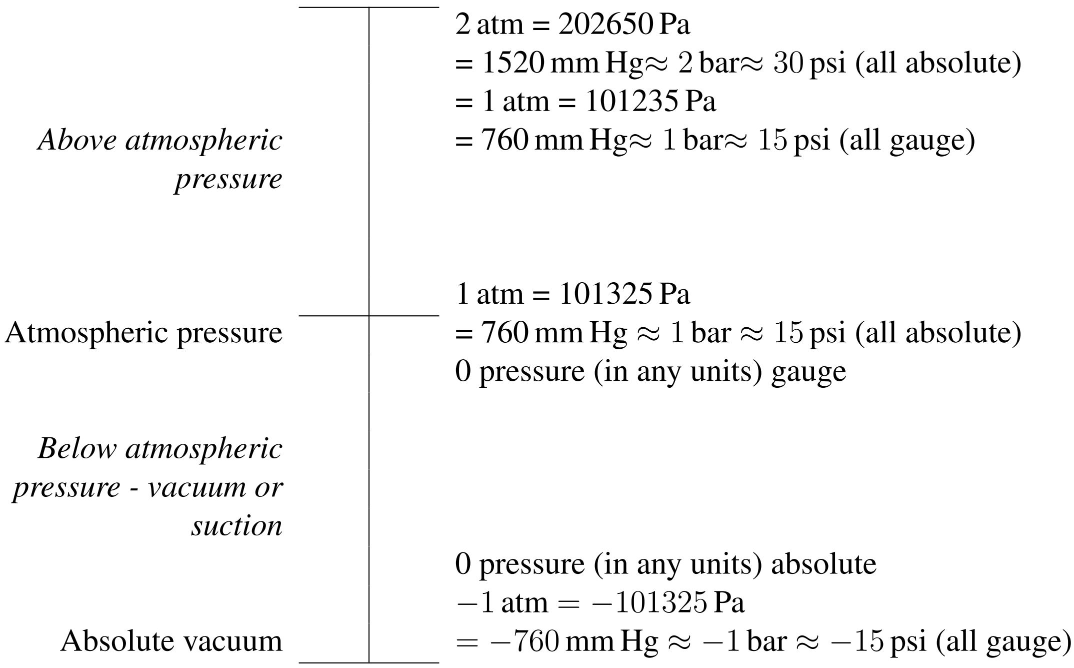


Figure 2.2: Relationships between gauge and absolute pressures, and between different units used to report pressures.

2.3.2 Variation of Pressure with Position in a Fluid at Rest

Consider an element of fluid at rest within a rectangular co-ordinate system defined by three axes, x, y, and z, as shown in Figure 2.3.

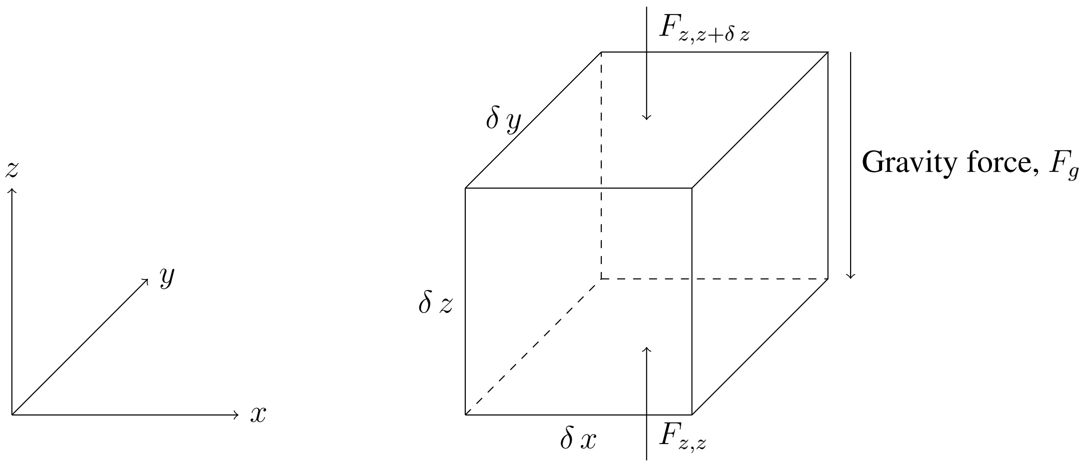


Figure 2.3: Rectangular co-ordinate system, and differential volume element.

The element is at rest (not moving), so there must be no overall (net) force acting on it – the various forces must be in balance. These forces include that arising from pressure acting downward on the top face of element, that arising from pressure acting upward on the bottom face of the element, and the (downward) force due to gravity acting on the mass of the element. (As there is no sideways movement, the sideways forces must also add up to a net force of zero; as there is no component in addition to the force arising from the pressure on each side face, these cancel out and are not included in this analysis). Just to clarify the nomenclature, $F_{z,z}$ means the force in the z-direction at the point z; $F_{z,z+\delta z}$ means the force in the z-direction at the point $z + \delta z$. The force in the z-direction at the point z is given by multiplying the pressure at that point, P_z, by the area over which the pressure operates, which is $\delta\,x \times \delta\,y$:

$$F_{z,z} = P_z \delta\,x \delta\,y \qquad \text{–upward on the bottom face} \tag{2.3.4}$$

Similarly, the force in the z-direction at the point $z + \delta\,z$ is given by

$$F_{z,z+\delta\,z} = P_{z+\delta\,z} \delta\,x.\delta\,y \qquad \text{–downward on the top face} \tag{2.3.5}$$

The "body force" (the downward force due to gravity) is given by (mass) $\times$ (acceleration due to gravity), where (mass) = (density $\times$ volume):

$$F_g = mg = \rho g \delta\,x.\delta\,y.\delta\,z \tag{2.3.6}$$

A force balance then gives:

$$\begin{aligned} F_{z,z+\delta\,z} + F_g &= F_{z,z} \\ P_{z+\delta\,z}\delta\,x.\delta\,y + \rho g \delta\,x.\delta\,y.\delta\,z &= P_z \delta\,x.\delta\,y \end{aligned} \tag{2.3.7}$$

Rearranging and dividing through by volume $(\delta\,x.\delta\,y.\delta\,z)$:

$$\begin{aligned} P_{z+\delta\,z}\delta\,x.\delta\,y - P_z \delta\,x.\delta\,y &= -\rho g \delta\,x.\delta\,y.\delta\,z \\ \frac{P_{z+\delta\,z} - P_z}{\delta\,z} &= -\rho g \end{aligned} \tag{2.3.8}$$

Setting the limit on the left hand side as $\delta\,z \to 0$ gives the definition of the derivative of P with respect to z:

$$\frac{\mathrm{d}\,P}{\mathrm{d}\,z} = -\rho g \tag{2.3.9}$$

In other words, the change in pressure, P, with height, z, is negative – pressure reduces with height (or increases with depth – as any diver knows). Then, to find the pressure at a particular position relative to the pressure at the surface, we integrate from the surface:

$$\int_{P_0}^{P} \mathrm{d}\,P = -\int_0^z \rho g \mathrm{d}\,z \tag{2.3.10}$$

If the density of the fluid is constant with z-position, and if g does not change with height (which it does not, over most changes in height of practical interest), then:

$$P - P_0 = -\rho g z \tag{2.3.11}$$

Note that z is negative as we go downwards – let's replace this with h, depth, which is positive as we go downwards, i.e. $h = -z$:

$$\begin{aligned} P - P_0 &= \rho g h \\ P &= P_0 + \rho g h \end{aligned} \tag{2.3.12}$$

Alternatively,

Hydrostatic Pressure

$$\Delta P = \rho g \Delta h \tag{2.3.13}$$

which we met previously, where ΔP is the difference in pressure between two points in a liquid separated by a horizontal distance Δh.

Note that this means that within a continuous expanse of the same fluid at rest, the pressure at any height is constant, or in other words, the pressure is the same across any horizontal plane.

Note that if $P_0 = 0$, for example, if we were considering pressure relative to the ambient atmosphere, and set atmospheric pressure equal to zero, then our equation would indicate a gauge pressure and would become:

$$P_{gauge} = \rho g h \tag{2.3.14}$$

2.4 Archimedes' Principle

Archimedes' principle states that the buoyant force on a submerged object is equal to the weight of the fluid it displaces. Our understanding of the relationship between pressure and depth can be used to demonstrate this principle, as shown in Figure 2.4.

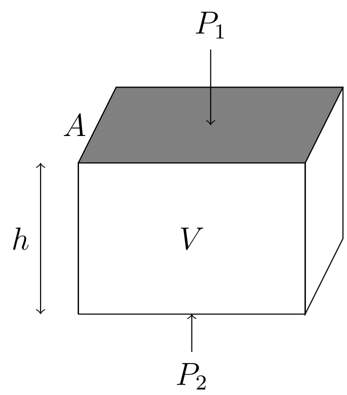


Figure 2.4: Archimedes' Principle.

The pressure on the top of the object is P_1, and the pressure on the bottom is P_2. This means that if the top and bottom of the object each have area A, the force on the top is $P_1 A$ downward, while the force on the bottom is $P_2 A$ upwards. The net force upwards on the object from the fluid is then,

$$F = P_2 A - P_1 A = (P_2 - P_1)\, A \tag{2.4.1}$$

If the height of the object is h, then from equation 2.3.12 we get $P_2 = P_1 + \rho_f g h$, so that,

$$F = (P_1 + \rho_f g h - P_1)\, A = \rho_f g h A = \rho_f g V = m_f g \tag{2.4.2}$$

where m_f is the mass of the fluid which is displaced by the object (not the mass of the object). This is Archimedes' Principle and the same goes for any shape of object. Thus the buoyancy force is given by the weight of the fluid displaced,

Archimedes' Principle

$$F_b = \rho_f g V_{\text{disp}} \tag{2.4.3}$$

The force due to gravity on the object is the object's mass times gravity,

$$F_g = mg = \rho g V \tag{2.4.4}$$

The net force on the object must be zero if it is to be a situation of fluid statics such that Archimedes principle is applicable, and is thus the difference of the object's weight and the buoyancy force,

$$F_T = 0 = F_g - F_b = mg - \rho_f g V_{\text{disp}} \tag{2.4.5}$$

If the buoyancy of an (unrestrained and unpowered) object exceeds its weight, it tends to rise, an object whose weight exceeds its buoyancy tends to sink.

2.5 Pascal's Principle

There is one big difference between liquids and gases. The density of a gas is easy to change. However, liquids are usually incompressible. Incompressiblility means that the density of a liquid is independent of the pressure. This is not perfectly true: liquids do contract and expand a little, but not much at all: this expansion and contraction can easily be neglected. We've already used fluid incompressibility; for example, the formula for how the pressure depends on the depth of the fluid assumed that the density remained constant, even though the pressure increases. Pascal's Principle also depends on it. Pascal's principle says that if you push at one end of the liquid, the pressure increases everywhere. If the liquid were compressible, what would just happen is that part of the liquid would become more dense. This is what happens to a solid (or the solid just moves). A gas, on the other hand, will compress uniformly. Stated precisely, Pascal's principle is:

Pascal's Principle

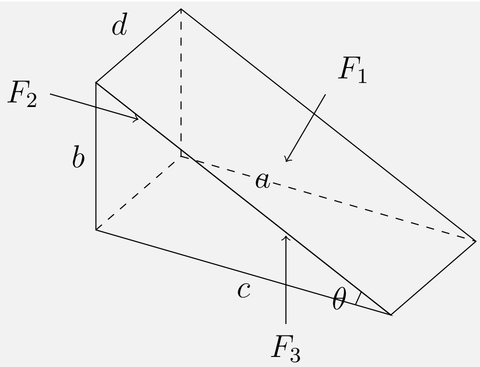
Any change in the pressure applied to a completely enclosed fluid is transmitted undiminished to all parts of the fluid and hence to the walls. Thus,

$$P_1 = P_2 = P_3$$

The proof of this can be seen in Derivation 2.1.

Derivation 2.1: Pascal's Principle.

Let us imagine a arbitrary right angled prismatic triangle in a fluid of density ρ,



This prismatic triangle is very small such that every part can be considered at the same depth, thus the effect of gravity is the same at all points.
The pressure of the liquid exerts the force normal to the surface it is acting on, thus,

$$F_1 = P_1 ad$$
$$F_2 = P_2 bd$$
$$F_3 = P_3 cd$$

Since the prism is in equilibrium, the net force in each direction (horizontal and vertical) is zero, thus,

$$F_1 \sin\theta = F_2$$
$$F_1 \cos\theta = F_3$$

From trigonometry we also know that $\sin\theta = b/a$ and $\cos\theta = c/a$, thus for horizontal force we get,

$$F_1 \sin\theta = F_2$$
$$P_1 ad\left(\frac{b}{a}\right) = P_2 bd$$
$$P_1 = P_2$$

and for vertical force,

$$F_1 \cos\theta = F_3$$
$$P_1 ad\left(\frac{c}{a}\right) = P_3 cd$$
$$P_1 = P_3$$

Thus this means that,

$$P_1 = P_2 = P_3$$

and Pascal's principle is proved.

This is the principle of how a hydraulic pump works, to say lift your car up at the garage. You have a two pistons, one narrower than than the other, each pushing on the same fluid. The wider one supports the car, as in Figure 2.5.

The pressure inside the fluid always obeys equation 2.3.13, so that,

$$P_2 = P_1 + \rho g\left(h_1 - h_2\right) \tag{2.5.1}$$

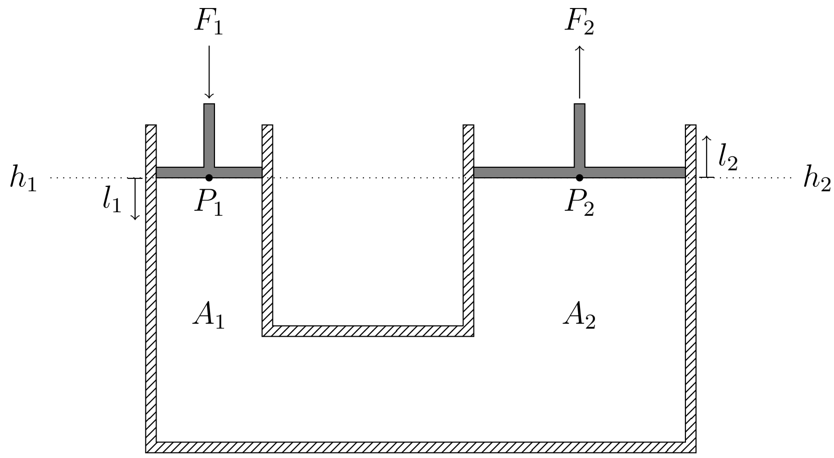


Figure 2.5: A simple hydraulic pump.

Say the two pistons are at the same height, then $P_1 = P_2$. Now say one piston is pushed down with a force F_1, thus increasing the pressure P_1. Thus P_2 increases as well, so that from equation 2.3.1,

$$P_1 = \frac{F_1}{A_1} = P_2 = \frac{F_2}{A_2} \tag{2.5.2}$$

where A_1 and A_2 are the areas of the two pistons. Thus,

$$F_2 = F_1 \frac{A_2}{A_1} \tag{2.5.3}$$

If $A_2 > A_1$, then the force at the second piston is larger than that at the other. This is why you put the car on the larger piston; the force you apply is magnified. If the two pistons aren't at the same height, you need to take into account the fact that the pressure at the two places is different from equation 2.5.1.

A hydraulic pump can magnify a force as much as you want; however, you don't get something for nothing. Remember that work is force times distance and that applied is that done, so,

$$W_1 = F_1 l_1 = W_2 = F_2 l_2 \tag{2.5.4}$$

So from equation 2.5.3,

$$l_2 = l_1 \frac{F_1}{F_2} = l_1 \frac{A_1}{A_2} \tag{2.5.5}$$

2.6 Pressure Measurement Devices

In practice, pressure is always measured by measuring a pressure difference. If the difference is between that of the fluid in question and that of a vacuum, then we are measuring the absolute pressure of the fluid. If, as is more usual, the difference is between the pressure of the fluid and that of the surrounding atmosphere, then this is a gauge pressure (so called because that's what pressure gauges normally measure). In this section we will consider several common devices for measuring pressure: the barometer, the manometer, refinements of the manometer for measuring small pressure differences more accurately, and the Bourdon gauge.

2.6.1 The Barometer

Figure 2.6 illustrates the basis of operation of a barometer [1]. The sealed tube contains a suitable liquid such as mercury (which has a high density of around $13560\,\mathrm{kg\,m^{-3}}$ and so allows short tubes to be used, and has a low vapour pressure, such that the space at the top of the tube is very close to a perfect vacuum). With the presence of a vacuum in the top of the sealed tube, the height of the column depends on the external pressure; to put it another way, the external pressure can only push the liquid column up so high and no higher.

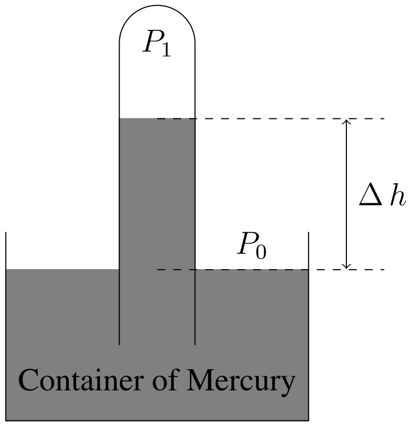


Figure 2.6: Diagram of a barometer, showing how the external atmospheric pressure forces liquid up the tube.

The difference in pressure between the base of the column of liquid and the top surface is (atmospheric − absolute vacuum), and the difference in height is given by the equation we derived above,

$$P_0 = \cancelto{0}{P_1} + \rho g \Delta h \tag{2.6.1}$$

For standard atmospheric pressure, the height of the column can be calculated in SI units as:

$$\Delta h = \frac{\Delta P}{\rho g} = \frac{101325}{13560 \times 9.81} = 0.762\,\mathrm{m}$$

If water were used, with a density of $1000\,\mathrm{kg\,m^{-3}}$, clearly the equivalent height would be 10.33 m, if a perfect vacuum could be maintained in the space at the top of the tube. But unlike mercury, water has a significant vapour pressure, so that at 15 °C, for example, the height would be about 0.18 m less than this.

Clearly, the actual height of the column of mercury in a barometer varies with the ambient pressure, and also to some extent with other weather conditions (in particular temperature, which causes thermal expansion of both the mercury and the casing).

2.6.2 Manometers

A manometer is a device in which a column of a suitable liquid is arranged such that the height difference between its two ends indicates the difference in pressure at the two ends. Figure 2.7 illustrates this.

At equilibrium, the difference in height indicates the difference between the two pressures. The pressure at the base of the manometer must be the same otherwise there would be flow

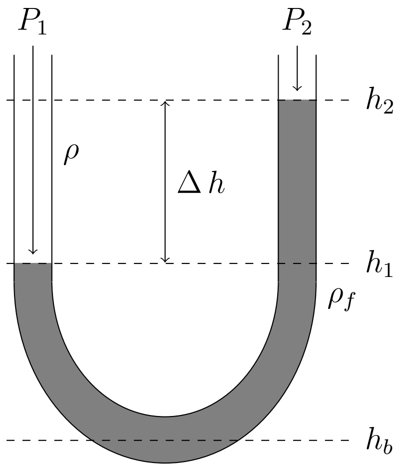


Figure 2.7: A U-tube manometer, in which the higher pressure P_1 pushes the column of liquid towards the lower pressure P_2.

to/away from the base (see Chapter 4). This means that,

$$\begin{aligned}
&\text{LHS: } P_1 + \rho g \left(h_2 - h_1\right) + \rho_f g \left(h_1 - h_b\right) = P_2 + \rho_f g \left(h_2 - h_1\right) && \text{:RHS} \\
&\text{LHS: } P_1 + \rho g \left(h_2 - h_1\right) + \rho_f g \left(h_1 - h_b\right) = P_2 + \rho_f g \left(h_2 - h_1\right) + \rho_f g \left(h_1 - h_b\right) && \text{:RHS} \\
&\text{LHS: } P_1 + \rho g \left(h_2 - h_1\right) = P_2 + \rho_f g \left(h_2 - h_1\right) && \text{:RHS}
\end{aligned}$$

as we know the pressure variation with fluid height from equation 2.3.13. This can be rearranged as,

$$P_1 - P_2 = \left(\rho_f - \rho\right) g \left(h_2 - h_1\right) \tag{2.6.2}$$

or in terms of the change in manometer fluid height, Δh,

Manometer

$$\Delta P = \left(\rho_f - \rho\right) g \Delta h \tag{2.6.3}$$

If one of the ends of the manometer is open to the atmosphere, then the pressure difference indicates the gauge pressure at the other end of the column of liquid. Alternatively, neither end may be at atmospheric pressure, as in the use of a manometer to relate pressure differences along a pipe to fluid flowrates, which we'll use in Chapter 5.

In order to improve the sensitivity of manometers, several refinements have been introduced. The simplest is the inclined manometer, in which the manometer is simply inclined at an angle θ to the horizontal. This does not alter the height difference at all, but it means this height difference is now extended over a longer length (given by $l = h/\sin\theta$), such that changes in length, and thus changes in pressure, can be measured more precisely.

Alternatively, the sensitivity of a manometer can be improved by using an intermediate liquid with its top surface in a vessel of wide cross sectional area compared with that of the U-tube (so the height of the intermediate fluid surface stays approximately constant), to amplify the change in heights produced by a small pressure difference. Figure 2.8 illustrates one arrangement that would achieve this.

Assume that the two pressures P_1 and P_2 act at the surface indicated by h_3 (i.e. P_1 and P_2 don't vary much with height, as would be the case for a "light" fluid of very low density ρ_1

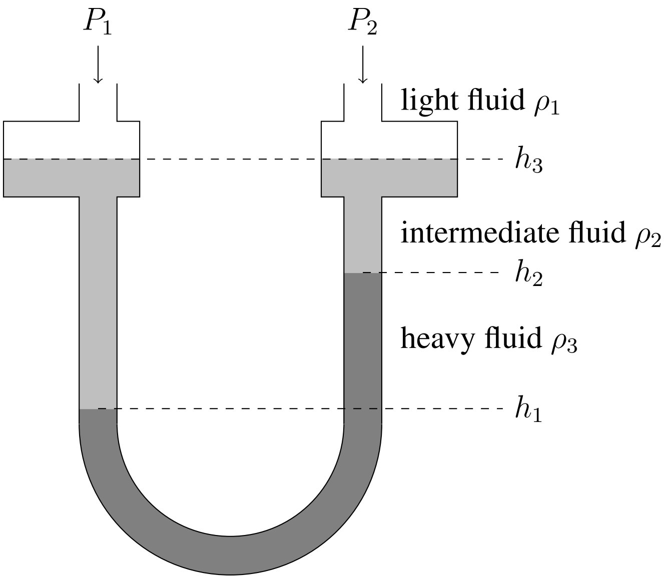


Figure 2.8: A compound manometer, $\rho_3 > \rho_2 > \rho_1$.The sensitivity is $\times 5$ or more if $\rho_2 \sim \rho_3$.

such as air), and that differences in h_3 are very small because of the large cross sectional area of this part of the manometer. The pressure at position h_1 on both sides of the U-tube must be the same. On the left hand side this is given by P_1 plus the contribution of the column of liquid of density ρ_2. This is equal to the pressure at position h_1 on the right hand side of the column, which is given by P_2 plus the contribution of the column of liquid of density ρ_2 and the contribution of the column of liquid of density ρ_3,

$$\text{LHS: } P_1 + \rho_2 g\,(h_3 - h_1) = P_2 + \rho_2 g\,(h_3 - h_2) + \rho_3 g\,(h_2 - h_1) \text{ :RHS}$$

Now we can rearrange this to relate the pressure difference between P_1 and P_2 to the various heights:

$$\begin{aligned} P_1 - P_2 &= \rho_2 g\,[(h_3 - h_2) - (h_3 - h_1)] + \rho_3 g\,(h_2 - h_1) \\ &= -\rho_2 g\,(h_2 - h_1) + \rho_3 g\,(h_2 - h_1) \\ &= (\rho_3 - \rho_2)\,g\,(h_2 - h_1) \end{aligned} \tag{2.6.4}$$

Clearly ρ_3 must be greater than ρ_2, otherwise it would float rather than stay in the bottom of the manometer. However, if ρ_2 and ρ_3 are close in value such that $(\rho_3 - \rho_2)$ is small, then a small difference in pressure will correspond to a large difference in height, $(h_2 - h_1)$. This becomes clearer if we rearrange the equation to express it in terms of the height difference resulting from a given pressure difference:

Compound Manometer

$$\Delta h = \frac{\Delta P}{(\rho_3 - \rho_2)\,g} \tag{2.6.5}$$

The difference in height for a given ΔP is larger than would be obtained if $\rho_2 = 0$, which is essentially the case in the ordinary manometer in which the "intermediate" fluid is air.

2.6.3 The Bourdon Gauge

The most common type of pressure gauge is the Bourdon gauge, which is compact, robust and easy to use, and which relies on the deformation of an elastic solid as its basis for indicating pressure. It comprises a curved tube of elliptical cross section closed at one end and free to move at this end, while the other end is fixed in position but open to entry of the fluid whose pressure is being measured. When the pressure exceeds the external pressure, the tube cross section tends to become more circular, causing the tube to uncurl slightly. The movement of the free end of the tube is transmitted to a scale via a suitable pointer and mechanical linkage, as illustrated in Figure 2.9.

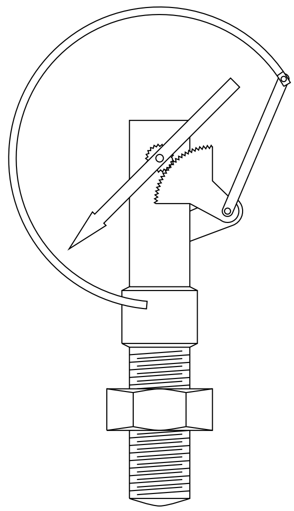


Figure 2.9: Diagram of a Bourdon gauge.

2.7 Pressure Heads

It will be clear from the above examples of pressure measurement that pressure is frequently measured in terms of the height of a column of liquid. In particular, the relationship between gauge pressure and height is directly proportional, such that pressure can in fact be visualised in terms of the height of a column of liquid of density ρ. This is termed the *pressure head*, a term used frequently in fluid flow, corresponding to

$$h = \frac{P}{\rho g} = \text{ the pressure head corresponding to } P \qquad (2.7.1)$$

2.8 References

[1] Chisholm, H., ed. [1911], *Encyclopaedia Britannica*, Cambridge University Press.

Chapter 3

Fluids and Flows

Contents

3.1 Chapter 3 ILOs

ILO 3.1. Determine what causes a fluid to flow.

ILO 3.2. Characterise the key regimes of flowing fluids.

ILO 3.3. Compare viscosity types of different fluids.

ILO 3.3. Derive the equation for the velocity profile in fully developed flow.

3.2 Introduction

Flow of fluids is always associated with pressure differences. As pressure is a form of energy, and energy can be converted into other forms, the relationship between pressure and flow is in some ways not always obvious or intuitive. One form of energy is thermal energy (heat), such that a flowing fluid converts some of its energy into heat. This is due to the internal friction of the flowing fluid, i.e. its viscosity. In this chapter we will firstly introduce the different types of flow or "flow regimes". As we'll see, the nature of fluid flow depends, among other things, on the fluid's viscosity.

3.3 Flow Regimes and Reynolds Number

Consider two tanks connected by a pipe, as shown in Figure 3.1. Tank A is at higher pressure (in this case atmospheric pressure), while Tank B is kept at a lower pressure through the action of a pump. Because there is a pressure difference between the tanks, fluid flows along the pipe from Tank A to Tank B at a velocity u (we will explore this relationship more later).

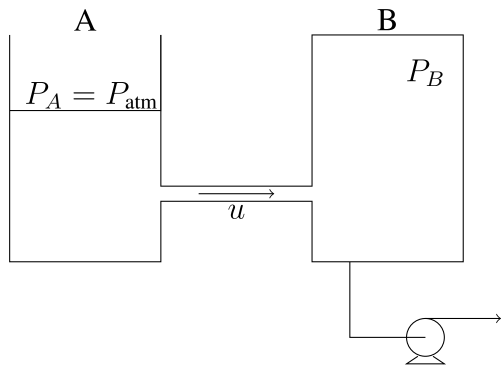


Figure 3.1: Fluid flow in a pipe between tanks at different pressures.

The effect of pressure drop on flow velocity depends on the type of flow patterns in the pipe and the flow regime. Figure 3.2 illustrates possible relationships between u and $(P_A - P_B)$ and defines different regimes corresponding to different patterns of flow.

From this we can make a number of observations:

- Gases can achieve much higher flow velocities than liquids; $u_{\text{gases}} >> u_{\text{liquids}}$.
- Region I corresponds to laminar incompressible flow, in which flow velocity is proportional to pressure drop; $u \approx \Delta P$. Both gases and liquids can undergo Region I flow.
- Region II corresponds to turbulent incompressible flow, in which $u \approx \Delta P^{0.5}$. Both gases and liquids can undergo Region II flow.
- Region III corresponds to compressible flow. This applies to gases only, and indicates that the pressure change is sufficiently large that the density of the gas changes significantly as it flows and can no longer be considered constant. In this region, $u \approx \Delta P^n$ where $n < 0.5$.

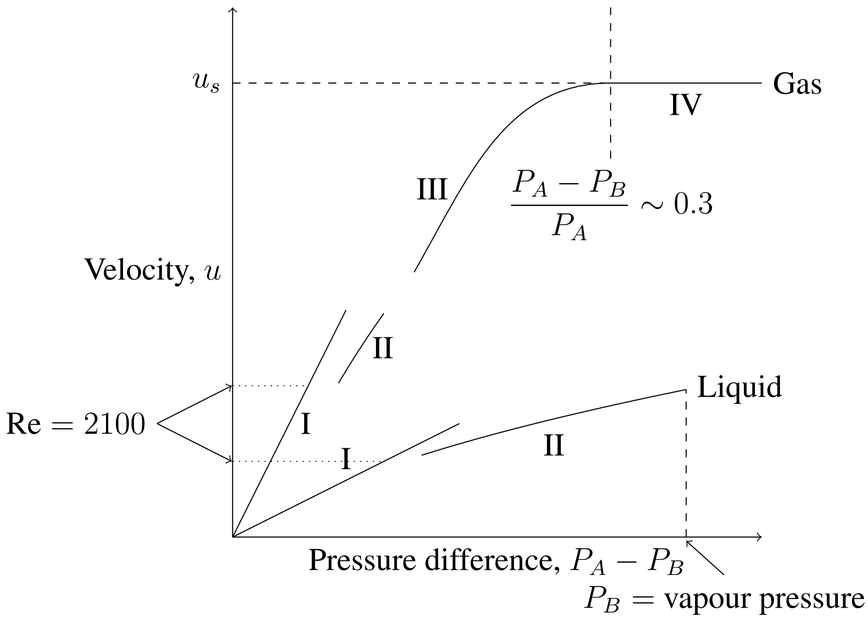


Figure 3.2: Velocity of fluid flow versus pressure difference.

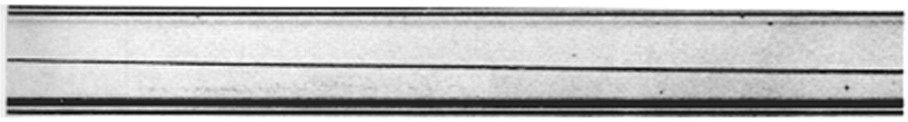
- Region IV applies to gases only and describes choking flow. This occurs when the gas is flowing at the speed of sound, u_s, and cannot flow any faster, no matter what size of pressure drop is applied. Thus under choking flow, u is independent of ΔP. Choking flow typically occurs when the magnitude of the pressure drop, $(P_A - P_B)$, is about 30% of the initial pressure of the gas.

Evidently there is a transition from laminar (Region I) flow to turbulent (Region II) flow – although it is not a sudden or clear transition. The transition from laminar to turbulent flow was first studied in the early 1880s by Osborne Reynolds [3], Professor of Engineering at the University of Manchester.

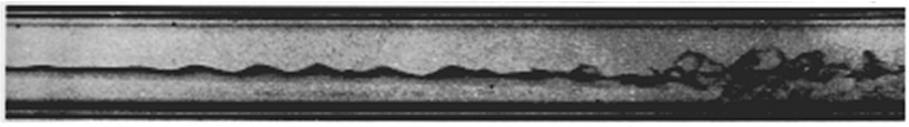
Reynolds constructed an apparatus in which the flowrate of water through a glass tube could be carefully controlled. He then introduced a fine stream of dye into the flow and observed the patterns of motion of the stream of dye. Figure 3.3 shows some images of his observations. At low flowrates (Figure 3.3(a)), the dye filament remained steady and unbroken as it flowed along with the water, indicating laminar flow. At higher flowrates ((b) and (c) in Figure 3.3), the flow became turbulent such that the dye filament began to oscillate and eventually broke up into eddies which dispersed the dye right across the tube. These experiments clearly demonstrated the transition from laminar to turbulent flow, and allowed the conditions leading to either laminar or turbulent flow to be identified.

Experiments with a variety of fluids of varying physical properties and of a range of pipe diameters and flowrates demonstrated that the nature of the flow depended on the fluid's density, ρ, and viscosity, μ, its average velocity through the pipe ($\bar{u}$, but we tend just to write u, as it's more convenient), and the pipe diameter, D. Larger pipes, faster velocities and greater densities tend to give more inertia to the flow and make it turbulent, while small pipes, slow flow and high viscosities tend to give laminar flows.

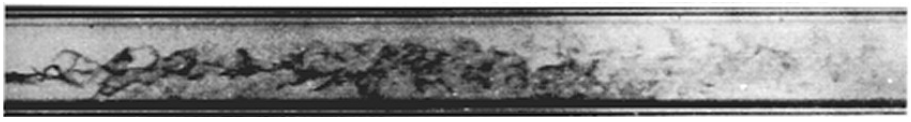
It was discovered that these parameters could be arranged to form a dimensionless group, which was later called the Reynolds Number, Re, in honour of Osborne Reynolds and his seminal work. The Reynolds Number determines the nature of the fluid flow and is defined as,



(a) Laminar Flow



(b) Transitional Flow



(c) Turbulent Flow

Figure 3.3: Flow patterns observed on injection of a stream of dye into a flowing fluid.Water (a) velocity $11\,\mathrm{cm\,s^{-1}}$, Re $= 1.5 \times 10^3$, (b) velocity $17\,\mathrm{cm\,s^{-1}}$, Re $= 2.34 \times 10^3$, and (c) velocity $54\,\mathrm{cm\,s^{-1}}$, Re $= 7.5 \times 10^3$ pipe ID 14 mm, dye injection method.

Reynolds Number

$$\text{Re} = \frac{Du\rho}{\mu} \tag{3.3.1}$$

The Reynolds Number is an example of a dimensionless number,

$$\frac{\mathrm{m} \times \frac{\mathrm{m}}{\mathrm{s}} \times \frac{\mathrm{kg}}{\mathrm{m}^3}}{\frac{\mathrm{kg}}{\mathrm{m\,s}}} = \text{no units}$$

Dimensionless numbers have the great benefit that they can bring together a wide range of experimental conditions and allow them to be described in simple terms, and that they allow similarities of behaviour at different scales to be identified. In this case, while many different conditions of fluid flow might exist, in terms of the density and viscosity of the fluid and the velocity and geometry of the flow, all these conditions can be made comparable by describing them in terms of the Reynolds Number. Dimensionless numbers frequently achieve their dimensionless status by representing the ratio of two things.

The Reynolds Number, the ratio is between inertial forces, which favour turbulent flow, and viscous forces, which favour laminar flow,

$$\text{Re} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}} = \frac{\rho u^2}{\mu u/D} = \frac{Du\rho}{\mu}$$

Based on his studies Reynolds found that:

Flow Regimes

- Re < 2100 is laminar flow,
- $2100 <$ Re < 10000, the flow is transitional between laminar and turbulent,
- Re $> 10,000$ is turbulent flow.

3.4 Viscosity

3.4.1 The Basis of Viscosity

When a force is applied to a body it deforms in some way. We are already familiar with two of these deformations, both of which are classed as volumetric strains, i.e. a uniform force is applied to every surface of a body, such as pressure, which causes the volume to decrease, compression, or increase, dilatation.

If a non-uniform force is applied, e.g. to just one side of a body, then we alter the shape, but not the volume, of the body. This is called a shearing strain. For example, a fluid can be between two parallel plates, if the lower plate is held fixed while the upper plate is moved parallel to the lower plate the body will then be sheared, This can be seen in Figure 3.4.

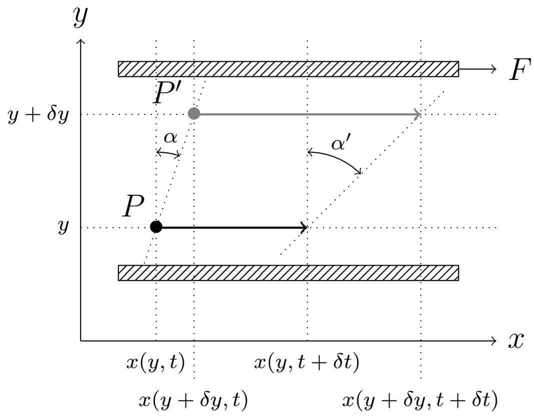


Figure 3.4: Analysis of simple shear.

Due to the force on the top plate, point P' will have moved slightly more than point P by a given time t. We can define the shear rate (rate of the sheaaring strain) in terms of the different velocities at the two positions as,

$$\dot{\gamma}\,(P,t) = \frac{\partial\, u}{\partial\, y} \tag{3.4.1}$$

Where u is the velocity in the x-direction of point P at time t. The full derivation of this can be seen in Derivation 3.1.

Derivation 3.1: Shear Rate.

We can define a shearing strain at a given time t as the tangent of angle α,

$$\gamma\left(P,t\right)=\frac{x\left(y+\delta y,t\right)-x\left(y,t\right)}{\left(y+\delta y\right)-y}=\left.\frac{\partial\, x}{\partial\, y}\right|_{t}$$

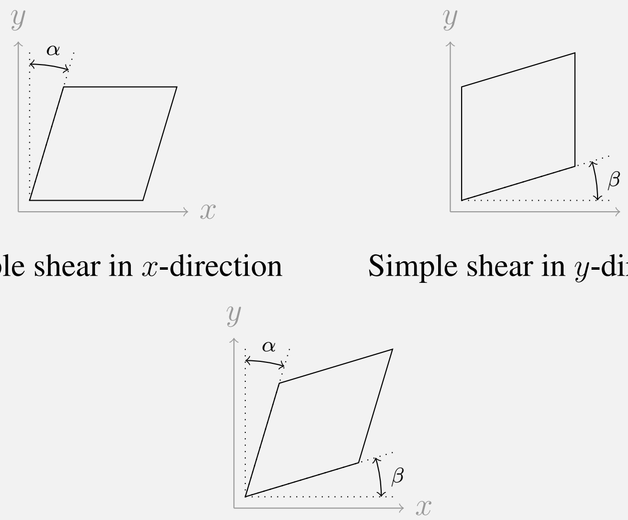
At a time $t+\delta t$, both points will have moved slightly, so the shearing strain at point P is now the tangent of angle α',

$$\gamma\left(P,t+\delta t\right)=\frac{x\left(y+\delta y,t+\delta t\right)-x\left(y,t+\delta t\right)}{\left(y+\delta y\right)-y}=\left.\frac{\partial\, x}{\partial\, y}\right|_{t+\delta t}$$

In expressing the shearing strain at time $t+\delta t$, we have been careful to follow the path of point P as it is a function of not only the time but of the point which is being considered. The rate of the shearing strain (typically called shear rate) is then the change of the shearing strain with time,

$$\begin{aligned}\dot{\gamma}\left(P,t\right)&=\frac{\gamma\left(P,t+\delta t\right)-\gamma\left(P,t\right)}{\left(t+\delta t\right)-t}\\&=\frac{\left.\frac{\partial\, x}{\partial\, y}\right|_{t+\delta t}-\left.\frac{\partial\, x}{\partial\, y}\right|_{t}}{\delta t}\\&=\frac{\partial}{\partial\, t}\left(\frac{\partial\, x}{\partial\, y}\right)=\frac{\partial}{\partial\, y}\left(\frac{\partial\, x}{\partial\, t}\right)\\&=\frac{\partial\, u}{\partial\, y}\end{aligned}$$

Where u is the velocity in the x-direction of point P at time t.
This expression is for a one-dimensional force producing a one-dimensional shearing strain, if the force is multi-dimensional then we can consider these together as in the figure below. This gives a rate of shearing strain as [2],



Simple shear in x-direction Simple shear in y-direction

Superposition of the two simple shears

$$\dot{\gamma}\left(P,t\right)=\frac{\partial\, u}{\partial\, y}+\frac{\partial\, v}{\partial\, x}$$

where v is the velocity in the y-direction of point P at time t. This can be generalised in n-dimensions for a non-compressible fluid as [1],

$$\dot{\boldsymbol{\gamma}} = \nabla \mathbf{V} + (\nabla \mathbf{V})^T \tag{3.4.2}$$

where $\nabla \mathbf{V}$ is the gradient of the velocity field (see SectionA.2.1).

The shear stress, τ, is equal to the applied force over the area,

$$\tau = \frac{F}{A} \tag{3.4.3}$$

The shear rate is then a function of the shear stress as,

$$\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}(\boldsymbol{\tau}) \tag{3.4.4}$$

The form of this function is then the viscosity, or viscosity function of the fluid. The viscosity is a property of the fluid by which it resists the motion from the shear stress, as there is this resistance it means that there is always a force needed for a fluid to flow. The viscosity is due to friction between the fluid particles on the molecular level; as there is a friction it means that some energy will be lost in the form of heat.

3.4.2 Types of Viscosity

The viscosity of a fluid and are broadly divided into three categories,

1. Newtonian
2. Time independent Non-Newtonian
3. Time dependent Non-Newtonian

When the viscosity of a liquid remains constant and is independent of the applied shear stress, such a liquid is termed a Newtonian liquid. This means that the function between the shear rate and shear stress is then a simple constant as,

Newtonian Viscosity

$$\tau = \mu \dot{\gamma} \tag{3.4.5}$$

Examples of Newtonian fluids include water, air, alcohol, and glycerol.

In the case of the non-Newtonian liquids, viscosity depends on the applied shear force and time. For time independent non-Newtonian fluid, when the shear rate is varied, the shear stress does not vary proportionally and is shown in Figure 3.5. The most common types of time independent non-Newtonian liquids include,

Psuedoplastic , this type of fluid displays a decreasing viscosity with an increasing shear rate is sometimes called shear-thinning. Examples include cream, ketchup, molasses, blood, and sand in water.

Dilatant , this type of fluid displays increasing viscosity with an increase in shear rate and is sometimes called shear-thickening. An example is suspensions of corn starch in water.

Bingham plastic , in this case a certain amount of force must be applied to the fluid before any flow is induced. Bingham plastic is a somewhat idealized representation of many real materials, for which the rate of shear is zero if the shearing stress is less than or equal to a yield stress, τ_y. Otherwise, it is directly proportional to the shearing stress in excess of the yield stress. Examples include clay suspensions, toothpaste, mayonnaise, chocolate, and mustard.

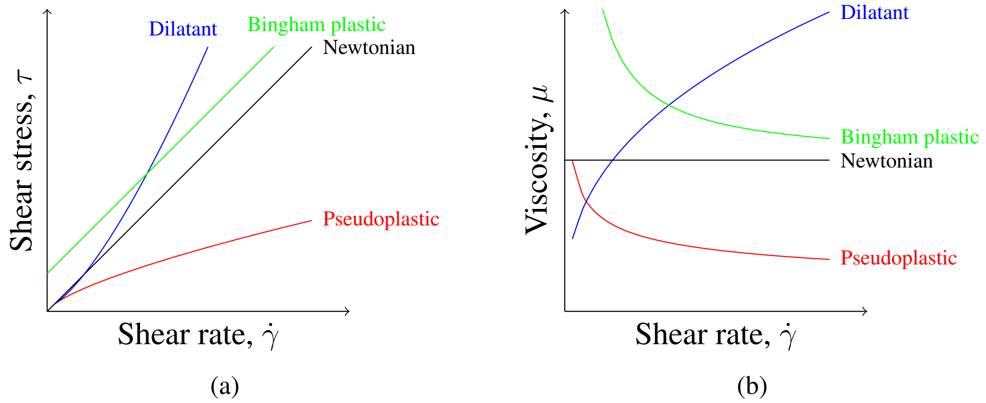


Figure 3.5: Various types of fluids based on viscosity, (a) shear stress vs shearrate and (b) viscosity vs shear rate.

Time dependent non-Newtonian fluids display a change in viscosity with time under conditions of constant shear rate. One type of fluid called Thixotropic undergoes a decrease in viscosity with time as shown in Figure 3.6(a); examples include yogurt, peanut butter, pectin gels, and many paints. The other type of time dependent non-Newtonian fluid is called Rheopectic. The viscosity of rheopexic fluids increases with the time as it is sheared at a constant rate, Figure 3.6(b); examples include Synovial fluid, printer ink, and gypsum paste.

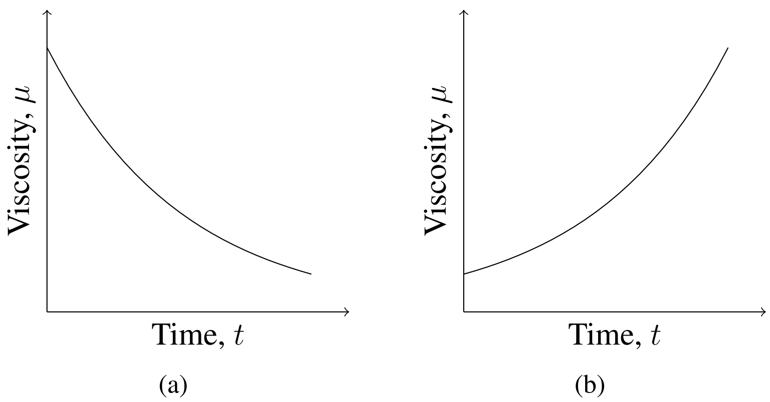


Figure 3.6: Change in Viscosity of time-dependent non-Newtonian fluids(a) Thixotropic fluid and (b) Rheopectic fluid.

3.5 Velocity Profiles

Let's consider the flow patterns of the fluid within the pipe linking Tanks A and B, Figure 3.1, with average velocities as per Figure 3.2. The overall average velocity is constant, but the velocity at different points in the pipe is likely to vary – for example, the velocity along the centre of the pipe is likely to be higher than that near the walls of the pipe, where the stationary walls slow the fluid down. So velocity depends on the radial position within the pipe, and across the pipe there is a velocity profile.

If the fluid had no viscosity, then there would be no friction to slow the fluid down, and all the elements of fluid would flow at the same velocity. Under this ideal, frictionless flow, $u\left(r\right) = \text{ constant } = \bar{u} = u_{\text{max}}$. A plot of velocity versus radial position would show constant velocities across the pipe, as illustrated below in Figure 3.7(a). This never occurs in reality, but it is approximated, as we'll see in a moment, by turbulent flow.

If the fluid has viscosity, then the shear experienced by the fluid at the wall is transmitted through the fluid, such that the fluid near the wall travels more slowly than the fluid in the centre. In fact, the fluid right at the pipe wall is considered to have zero velocity, while the velocity increases towards the centre of the pipe. In this case, there are two extreme possibilities for the nature of the fluid flow; laminar flow, and turbulent flow.

3.5.1 Laminar Flow

Under laminar flow, the flow is smooth, steady and ordered such that fluid elements flow along individual layers (laminae) or streamlines that do not cross each other. Flow is dominated by viscosity, leading to a parabolic flow profile as illustrated in Figure 3.7(b). Laminar flow occurs at low flow rates in which the fluid has little inertia.

It is possible to derive the velocity profile for a fluid under laminar flow conditions with knowledge of the velocity. Let us take a long, straight, constant diameter section of a pipe, which we can think of as a fully developed laminar flow. The gravitational effects (mass forces) will be neglected. The velocity profile is the same at any cross section of the pipe. Taking a cylindrical control volume, as in Figure 3.8 and applying a force balance to this we get a balance between pressure and viscous forces [1].

The pressure difference acting on the end of the cylinder of area πr^2 and the shear stress acting on the lateral surface of the cylinder of area $2\pi r L$ added must be equal to zero as the fluid is not accelerating. This force balance can be written as:

$$\begin{aligned} P\pi r^2 - \left(P - \Delta P\right)\pi r^2 - 2\pi r L\tau &= 0 \\ \Delta P\pi r^2 &= 2\pi r L\tau \\ \Delta P r &= 2L\tau \\ \tau &= \frac{r}{2}\frac{\Delta P}{L} \end{aligned} \tag{3.5.1}$$

Equation 3.5.1 represents the basic balance in forces needed to drive each fluid particle along the pipe with constant velocity. It is interesting to note that since ΔP is not a function of the radial coordinate, r , it follows that $2\tau/r$ must also be independent of r, i.e. a constant.

To carry the analysis further we must prescribe how the shear stress is related to the velocity of the fluid. This is the critical step that separates the analysis of laminar flow

[1] We will apply more rigorous balances in Chapters 4, 6, and 7

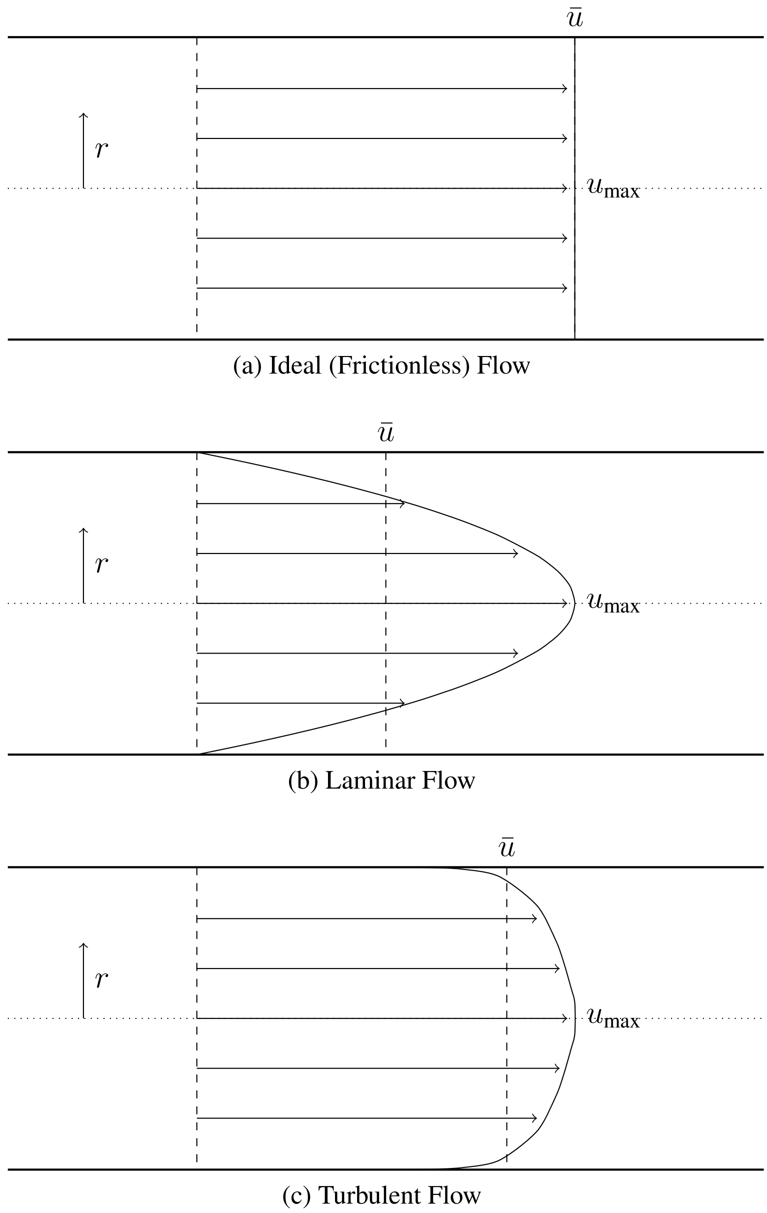


Figure 3.7: Flow profiles in pipes under (a) ideal, (b) laminar, and (c) turbulent flow.

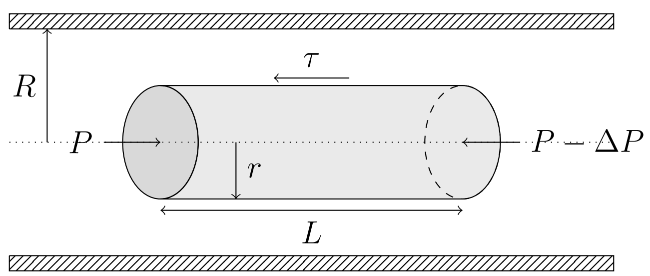


Figure 3.8: Cylindrical fluid element within a pipe.

from that of turbulent flow, i.e. the shear stress dependence for turbulent flow is very complex and has no current analytical solution. However, for laminar flow of a Newtonian fluid, the shear stress is simply proportional to the velocity gradient as in equation 3.4.5. In the notation of our example this becomes,

$$\tau = -\mu \frac{\mathrm{d}\, u}{\mathrm{d}\, r} \tag{3.5.2}$$

The negative sign is included as y increases as we move away from the outside pipe wall, whereas r increases as we move towards the outside pipe wall and away from the pipe center, thus $\mathrm{d}\,y = -\mathrm{d}\,r$. Combining this with equation 3.5.1 gives,

$$-\mu\frac{\mathrm{d}\,u}{\mathrm{d}\,r} = \frac{r}{2}\frac{\Delta P}{L}$$
$$\frac{\mathrm{d}\,u}{\mathrm{d}\,r} = -\frac{r}{2\mu}\frac{\Delta P}{L} \tag{3.5.3}$$

Because the fluid is viscous it sticks to the pipe wall so that $u = 0$ at $r = R$, this is called the no-slip boundary condition. Integrating under this condition gives,

$$\int_0^u \mathrm{d}\,u = -\frac{\Delta P}{L}\int_R^r \frac{r}{2\mu}\mathrm{d}\,r$$
$$[u]\,0^u = -\frac{\Delta P}{L}\left[\frac{r^2}{4\mu}\right]_R^r$$
$$u = -\frac{\Delta P}{L}\left(\frac{r^2}{4\mu} - \frac{R^2}{4\mu}\right)$$
$$u = \frac{1}{4\mu}\frac{\Delta P}{L}\left(R^2 - r^2\right) = \frac{\Delta P R^2}{4\mu L}\left(1 - \left(\frac{r}{R}\right)^2\right) \tag{3.5.4}$$

The volumetric flowrate, Q[1], through the pipe can be obtained by integrating the velocity profile across the pipe. Since the flow is axisymmetric about the center-line, the velocity is constant on small area elements consisting of rings of radius r and thickness $\delta\,r$. Thus,

$$\mathrm{d}\,Q = u\mathrm{d}\,A = u2\pi r\mathrm{d}\,r$$
$$\mathrm{d}\,Q = \frac{1}{4\mu}\frac{\Delta P}{L}\left(R^2 - r^2\right)2\pi r\mathrm{d}\,r$$
$$\int_0^Q \mathrm{d}\,Q = \frac{\pi}{2\mu}\frac{\Delta P}{L}\int_0^R \left(R^2 - r^2\right)r\mathrm{d}\,r$$
$$[Q]_0^Q = \frac{\pi}{2\mu}\frac{\Delta P}{L}\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^R$$
$$Q = \frac{\pi}{2\mu}\frac{\Delta P}{L}\frac{R^4}{4} = \frac{\pi R^4}{8\mu}\frac{\Delta P}{L} \tag{3.5.5}$$

This can be written in terms of the pipe diameter as,

Hagen-Poiseuille Equation

$$Q = \frac{\pi \Delta P D^4}{128\mu L} \tag{3.5.6}$$

[1]Ideally, we would write $\dot{Q}$, to underline that it is a flowrate and to be consistent with $\dot{M}$. However, $\dot{Q}$ is typically used to indicate heat transfer rate.

This means for horizontal pipe laminar flow the total flowrate is:

- directly proportional to the pressure drop
- inversely proportional to the viscosity
- inversely to the pipe length
- proportional to the pipe diameter to the fourth power

By definition, the average velocity is the total flowrate divided by the cross-sectional area of the pipe, as,

$$\bar{u} = \frac{Q}{A}$$
$$\bar{u} = \frac{\left(\frac{\pi D^4}{128\mu}\frac{\Delta P}{L}\right)}{\left(\frac{\pi D^2}{4}\right)}$$

Laminar Flow Average Velocity

$$\bar{u} = \frac{\Delta P D^2}{32\mu L} \tag{3.5.7}$$

The maximum velocity in the pipe is at the centre-line of the pipe, i.e. $r = 0$, thus from equation 3.5.4,

$$u_{\text{max}} = \frac{1}{4\mu}\frac{\Delta P}{L}\left(R^2 - (0)^2\right)$$
$$u_{\text{max}} = \frac{\Delta P R^2}{4\mu L} = \frac{\Delta P D^2}{16\mu L} \tag{3.5.8}$$

Comparing this with equation 3.5.7 gives,

$$u_{\text{max}} = 2\bar{u} \tag{3.5.9}$$

3.5.2 Turbulent Flow

Turbulent flow, by contrast to laminar flow, is wild and complex. Under turbulent flow, the paths of individual elements of fluid are no longer straight and steady, but are erratic and intercrossing. Thus, only the average motion of the fluid is in the direction of flow (parallel to the axis of the pipe). Thus the velocity profile illustrated in Figure 3.7(c) shows the average velocity at different radial positions.

Turbulent flow is found at high flow rates in which the fluid's inertia dominates over its viscosity in determining the flow patterns. Turbulent flow exhibits a high shear stress at the wall and steep velocity gradient near the wall. Under fully developed turbulent flow, the velocity profile away from the walls is quite flat, approaching that of ideal flow, with the average velocity in a circular pipe approximately 82% of the maximum velocity: $\bar{u} \approx 0.82 u_{\text{max}}$.

The fluid near the wall flows more slowly because the wall is stationary and the viscosity of the fluid transmits momentum between the stationary wall and the flowing fluid. Under turbulent flow, the layer near the wall, where the velocity profile is affected by the presence of the stationary surface, is called the laminar sublayer or viscous sublayer.

Unlike the laminar flow, it is not currently possible to develop an analytical solution to the velocity profile under turbulent flow conditions.

3.6 References

[1] Batchelor, G. K. [1967], *An Introduction to Fluid Dynamics*, Cambridge University Press.

[2] Fredrickson, A. G. [1964], *Principles and Applications of Rheology*, Prentice-Hall.

[3] Reynolds, O. [1883], 'An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels', *Philosophical Transactions of the Royal Society of London* **174**, 935–982.

Chapter 4

Flow Dynamics — Continuity Equations

Contents

4.1 Chapter 4 ILOs

ILO 4.1. Recognize the derivtion of the general continuity equation.

ILO 4.2. Calculate mass balances for fluid systems using the mass continuity equation.

ILO 4.3. Explain the derivation of Bernoulli's equation in terms of the momentum continuity balance.

ILO 4.4. Interpret the terms in Bernoulli's equation with respect to energy conservation.

4.2 The General Continuity Equation

A continuity equation (or conservation law) is an integral relation stating that the rate of change of some integrated property φ defined over a control volume Ω must be equal to the amount that is lost or gained through the boundaries Γ of the volume plus what is created or consumed by sources and sinks s inside the volume, Figure 4.1.

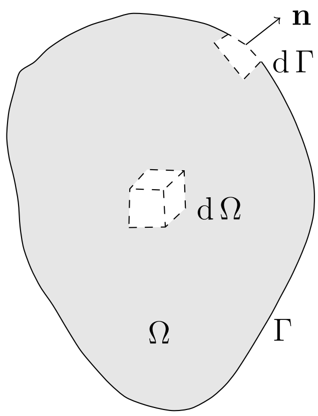


Figure 4.1: Control volume for the general continuity equation.

This is expressed mathematically by the following integral continuity equation,

$$\begin{aligned} \text{Accumulation} &= \text{(Input} - \text{Output)} &+& \text{(Generation} + \text{Consumption)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\oiiint_{\Omega} \varphi \,\mathrm{d}\,\Omega &= -\oiint_{\Gamma} \varphi \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}\,\Gamma &+& -\oiiint_{\Omega} s\,\mathrm{d}\,\Omega \end{aligned} \tag{4.2.1}$$

where $\mathbf{V}$ is the flow velocity of the fluid, $\mathbf{n}$ is the outward-pointing normal vector, and s is the sources and sinks in the flow, taking the sinks as positive. This expression can be rearranged to form the simplified non-integral general mass balance,

$$\frac{\partial \varphi}{\partial t} + \nabla\cdot(\varphi\mathbf{V}) + s = 0 \tag{4.2.2}$$

The full derivation can be seen in Derivation 4.1.

Derivation 4.1: General Continuity Equation.

Equation 4.2.1 contains both surface and volume integrals, therefore it is not that convenient for us to solve. We can use the divergence theorem (Section A.3.3) to transform the surface integral into a volume integral [5]. It states that the flow through a closed surface is related to the divergence (Section A.2.2) of the flow in the volume enclosed, such that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\oiiint_{\Omega} \varphi\,\mathrm{d}\,\Omega = -\oiiint_{\Omega} \nabla\cdot(\varphi\mathbf{V})\,\mathrm{d}\,\Omega - \oiiint_{\Omega} s\,\mathrm{d}\,\Omega$$

Applying the Reynolds transport theorem (Section A.3.4) [6] to the above equation produces,

$$\oiiint_{\Omega} \frac{\partial\varphi}{\partial t}\mathrm{d}\,\Omega = -\oiiint_{\Omega} \nabla\cdot(\varphi\mathbf{V})\,\mathrm{d}\,\Omega - \oiiint_{\Omega} s\,\mathrm{d}\,\Omega$$

Combining all terms into the volume integral gives,

$$\oiiint_{\Omega} \left(\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{V}) + s \right) \mathrm{d}\,\Omega = 0$$

This means that the integral must be zero for any control volume, this can only be true if the integrand itself is zero, so that we get the general form of the continuity equation,

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{V}) + s = 0$$

4.3 Mass Balance

The fact that (for practical, fluid flow purposes) mass is conserved means that what flows into one end of a pipe must flow out the other end and/or accumulate. Similarly, in other geometries (fluid flows in other contexts, just not in pipes), the flow of fluid must obey the law of conservation of mass. Expressing this mathematically is therefore an important part of fluid flow calculations.

When the property φ in equation 4.2.2 is taken as the mass per unit volume, the density ρ, then we generate the conservation of mass equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \qquad (4.3.1)$$

with a velocity vector $\mathbf{V}$[1]. The source and sink term s is taken here as 0 as we are assuming we are not creating or destroying mass.

Our mathematical description of this physical reality will be more accurate and powerful if we have a good understanding of the idea that fluids flow along streamlines.

4.3.1 Streamlines and Streamtubes

When fluids flow, the velocity over a plane at right angles to the flow is not normally uniform, as we saw in Figure 3.7. The variation of velocity can be shown by drawing streamlines, such that the velocity vector is always tangential to the streamline. Thus, by definition, there cannot be a net flow of fluid across a streamline (because the velocity vectors do not cross streamlines but run tangential to them). The flowrate between two streamlines must therefore always be constant (as fluid cannot enter or leave the space between the streamlines – there is no escape). If the fluid speeds up, we show this by making the streamlines closer together (such that the area between them stays constant). To put it another way, if a given quantity of fluid were flowing through a smaller pipe (equivalent to streamlines being closer together), it would need to flow faster. Figure 4.2 illustrates parallel streamlines in steady flow in a straight tube, and streamlines coming closer together as the fluid enters a constriction and speeds up.

In laminar flow, sometimes called streamline flow, the fluid flows along streamlines and does not cross them. In turbulent flow, circulations and eddies mean that fluid elements do cross streamlines, but there is no net flow across a streamline, and the average velocity is along the streamline.

[1] Velocity vector is cartesian coordinates (x, y, z), $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

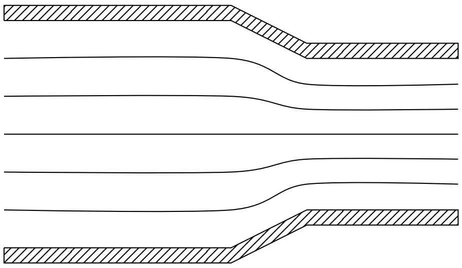


Figure 4.2: Streamlines in steady straight flow and approaching a constriction.

A group of streamlines can be visualised as forming a streamtube, and the whole area of flow can be visualised as bundles of streamtubes.

Consider flow of fluid through a streamtube as shown in Figure 4.3, in which the cross sectional area of the tube is sufficiently small that the fluid can be considered to have a uniform velocity across the tube. Fluid flows in at point 1 with a velocity u_1 through a cross sectional area A_1, and out at point 2 with velocity u_2 and cross sectional area A_2.

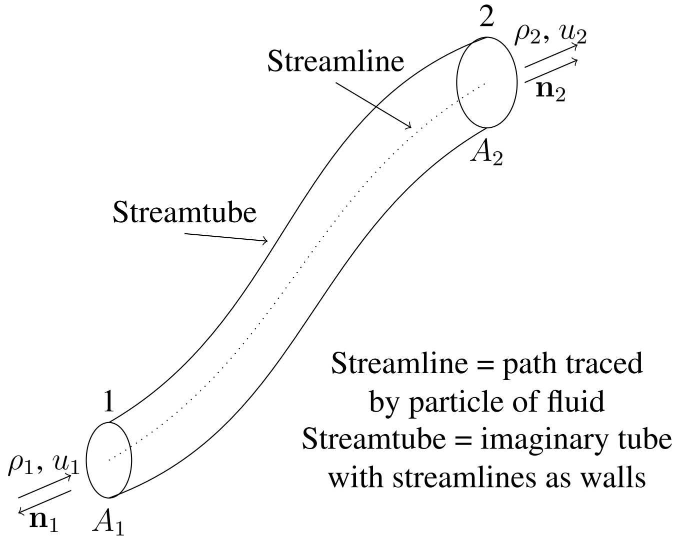


Figure 4.3: Flow through a streamtube.

4.3.2 Mass Balance in a Stream Tube

Let us now think about our mass continuity equation, equation 4.3.1 related to this stream tube. If our system is steady-state then nothing changes with time, this means that $\partial/\partial t = 0$, which produces,

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \nabla \cdot \rho \mathbf{V} = 0$$
$$\nabla \cdot \rho \mathbf{V} = 0 \tag{4.3.2}$$

To get our mass balance for the stream tube we can integrate this over the volume of the stream tube,

$$\oiiint_{\Omega} \nabla \cdot \rho \mathbf{V} \mathrm{d}\,\Omega = 0 \tag{4.3.3}$$

In this case we know what crosses the surface of the stream tube. This means that we can use the divergence theorem, as we did for the general continuity equation, to transform this back into a surface integral as,

$$\oiint_{\Gamma} \rho \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}\Gamma = 0 \tag{4.3.4}$$

Physically, this result implies that the influx of mass into the control volume must equal the efflux of mass at steady state. As our stream tube is defined with no flow through the sides, that is, $\mathbf{V} \cdot \mathbf{n}|_{\text{sides}} = 0$. There is flow only at the inlet (1) and exit (2). This means that the integral is solved to be,

$$\begin{aligned} \rho_1 u_1 (-1) A_1|_{\text{inlet}} + 0|_{\text{sides}} + \rho_2 u_2 (1) A_2|_{\text{outlet}} &= 0 \\ -\rho_1 u_1 A_1 + \rho_2 u_2 A_2 &= 0 \\ \rho_1 u_1 A_1 &= \rho_2 u_2 A_2 \end{aligned} \tag{4.3.5}$$

The mass flowrate, $\dot{M}$, is equal to the velocity times the area (the volumetric flowrate) times the density, which means that we get,

Mass Continuity Equation

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 = \dot{M} \tag{4.3.6}$$

This indicates that the mass flowrate must be constant along a streamtube, and that changes in the area of flow must be balanced by changes in the fluid velocity and/or density.

If density is constant (i.e. incompressible flow, as for a liquid or for a gas flowing under a small pressure drop), then

$$u_1 A_1 = u_2 A_2 = Q \tag{4.3.7}$$

where Q is the volumetric flowrate.

Averaging across an entire cross section comprising bundles of streamtubes, we need to use the average velocity at each point:

$$\bar{u}_1 A_1 = \bar{u}_2 A_2 \tag{4.3.8}$$

Frequently we just write u instead of $\bar{u}$; when we see u, we generally assume it means the average velocity, unless the context indicates otherwise.

The mass continuity can also be generalised to any number of inlet and outlets as,

$$\sum_{i=\text{inlets}} \rho_i u_i A_i = \sum_{j=\text{outlets}} \rho_j u_j A_j = \dot{M}_T \tag{4.3.9}$$

Understanding the requirement for continuity of flow can lead us to an alternative version of the Reynolds Number. Frequently we are given the mass flowrate of a fluid flowing through a pipe, from which we want to calculate the Reynolds Number in order to determine the nature of the flow. To do so, we must convert the mass flowrate, $\dot{M}$, to a volumetric flowrate, Q, via the fluid density, then calculate the fluid's average velocity from $Q = uA$. Thus,

$$u = \frac{Q}{A} = \frac{\dot{M}}{\rho A} \tag{4.3.10}$$

and

$$\text{Re} = \frac{Du\rho}{\mu} = \frac{D\dot{M}\rho}{\rho A\mu} = \frac{D\dot{M}}{\frac{\pi D^2}{4}\mu} = \frac{4\dot{M}}{\pi D\mu} \tag{4.3.11}$$

This version allows the Reynolds number to be calculated directly from the mass flow rate, without having to look up the density or to calculate the fluid's average velocity.

4.4 Momentum Balance

When the property φ in equation 4.2.2 is taken as the mass flux, or momentum density $\rho\mathbf{V}$ then we get[1],

$$\frac{\partial\,\rho\mathbf{V}}{\partial\,t} + \nabla\cdot(\rho\mathbf{V}\otimes\mathbf{V}) + s = 0 \tag{4.4.1}$$

The generic momentum source/sink s can be made specific by breaking it up into two new terms, one to describe stresses on our fluid element, $\boldsymbol{\sigma}$ made of the pressure (P) and shear stresses (τ), and one for any external body forces per unit mass on the fluid element, $\mathbf{f}$. This gives,

$$\rho\frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} = -\nabla P + \nabla\cdot\boldsymbol{\tau} + \rho\mathbf{f} \tag{4.4.2}$$

where $\mathrm{D}\,\mathbf{V}/\mathrm{D}\,t$ is defined as the substantial derivative which is taken as the differential in terms of time and space for a variable that varies in time and space (Section A.2.4). The full derivation of this equation can be seen in Derivation 4.2.

Derivation 4.2: General Navier Stokes Equation.

Equation 4.4.1 can be simplified, by chain rule and expansion, to become,

$$\mathbf{V}\frac{\partial\,\rho}{\partial\,t} + \rho\frac{\partial\,\mathbf{V}}{\partial\,t} + (\nabla\cdot(\rho\mathbf{V}))\,\mathbf{V} + \rho\,(\mathbf{V}\cdot\nabla)\,\mathbf{V} + s = 0$$
$$\mathbf{V}\left(\frac{\partial\,\rho}{\partial\,t} + \nabla\cdot(\rho\mathbf{V})\right) + \rho\left(\frac{\partial\,\mathbf{V}}{\partial\,t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\right) + s = 0$$

The left-hand expression enclosed in the parentheses we know from the mass continuity, equation 4.3.1, is equal to 0, thus,

$$\rho\left(\frac{\partial\,\mathbf{V}}{\partial\,t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\right) + s = 0$$
$$\rho\frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} + s = 0$$

where $\mathrm{D}\,\mathbf{V}/\mathrm{D}\,t$ is defined as the substantial derivative which is taken as the differential in terms of time and space for a variable that varies in time and space (Section A.2.4).

[1] $\otimes$ is the tensor product, in this case a dyad due to the fact that $\mathbf{V}$ is a vector

The generic momentum source/sink s can be made specific by breaking it up into two new terms, one to describe stresses on our fluid element, $\boldsymbol{\sigma}$, and one for any external body forces per unit mass on the fluid element, $\mathbf{f}$. This means we have,

$$\rho \frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}$$

The stress tensor, often called a Cauchy stress tensor [4], in 3 dimensions is given by,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

where σ are the normal stresses and τ the shear stresses. The first subscript refers to the surface normal to that direction and the second subscript refers to the direction of the force. The deviatoric stress tensor can be obtained by subtracting the hydrostatic stress tensor from the Cauchy stress tensor, i.e. the mechanical pressure,

$$\boldsymbol{\sigma} = -\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix} + \begin{pmatrix} \sigma_{xx} + P & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} + P & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} + P \end{pmatrix} = -P\mathbf{I} + \boldsymbol{\tau}$$

The motivation for doing this is that pressure is typically a variable of interest (or control), and also this simplifies application to specific fluids later on since the rightmost tensor $\boldsymbol{\tau}$ must be zero for any fluid at rest. The source term is now the divergence of the Cauchy stress tensor plus the external forces, so that the equation with the use of the stress tensor above becomes (Section A.2.1),

$$\rho \frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} = -\nabla P + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$

This is the Cauchy momentum equation or the most general form of the Navier-Stokes equation. This equation is still incomplete to allow us to solve our fluid systems. For completion, we must make hypotheses on the forms of τ, P and ρ, i.e. we need a constitutive law for the stress tensor which can be obtained for specific fluid families and an equation of state for the density.

4.4.1 Simplification to Bernoulli's Equation

The general Navier-Stokes equation, equation 4.4.2, can then be simplified for a number of conditions to allow use to produce an equation that is useful for many fluids flowing in a pipe. The key assumptions that we can use are,

Assumption 1 - Incompressible Newtonian fluid

For an ideal fluid we make the assumption that the fluid is incompressible ($\rho =$ constant) and Newtonian ($\mu =$ constant).

Assumption 2 - Inviscid Flow

To make our base case of the Bernoulli's equation the most simple, we can make the assumption of negligible viscosity [2], thus we lose the shear stress term in the equation.

We will look how to include some viscosity effects in Chapter 6.

Assumption 3 - Steady State

Under steady state conditions there is no variation in any of the flow properties with time, so that $\partial/\partial t = 0$.

Assumption 4 - Irrotational flow

Irrotational flow occurs where the curl (Section A.2.3) of the velocity (vorticity) of the fluid is zero everywhere, by definition $\nabla \times \mathbf{V} = 0$ [3]. This means that the circulation around any arbitrary loop in an irrotational flow pattern is zero. If a flow is inviscid and at any point in time the fluid is irrotational it will always be irrotational.

Assumption 5 - Gravity is the only external force

If the only external force acting on our system is gravity, and the upward direction is defined as the positive z-direction, then the external force on the element can be taken as just the negative of the gravity force, g, times by the z director.

Assumption 6 - One-Dimensional Flow

To satisfy the fact that the gradient equals zero, the integral must equal a constant, if we only have one-dimensional flow then the velocity vector can be simplifed as a constant, $\mathbf{V} = u$ (as in invicid flow the velocity profile is flat as in Figure 3.7(a)).

Due to these simplifications we can apply the general Navier-Stokes, equation 4.4.2, over a large distances, e.g. pipe, rather than a small fluid element. This then means that equation 4.4.2 becomes the much simplifed,

$$P + \frac{\rho u^2}{2} + \rho g h = \text{Constant} \tag{4.4.3}$$

The full derivation of which can be seen in Derivation 4.3. This can also be taken over two points in a system, 1 and 2, and divided through by ρg we get,

Bernoulli's Equation

$$\frac{P_1}{\rho_1 g} + \frac{u_1^2}{2g} + h_1 = \frac{P_2}{\rho_2 g} + \frac{u_2^2}{2g} + h_2 \tag{4.4.4}$$

Derivation 4.3: Bernoulli's Equation.

Assumption 1 - Incompressible Newtonian fluid

For an ideal fluid we make the assumption that the fluid is incompressible (ρ = constant) and Newtonian (μ = constant), this means that equation 4.2.2 can be simplified to become,

$$\frac{\partial \rho}{\partial t}\overset{0}{} + \rho \nabla \cdot \mathbf{V} = 0$$

$$\nabla \cdot \mathbf{V} = 0$$

For a Newtonian fluid, the stress is proportional to the rate of deformation (shear rate), as in equation 3.4.5 with the shear rate given in vector form by equation 3.4.2 [1],

$$\boldsymbol{\tau} = \mu \left(\nabla \mathbf{V} + (\nabla \mathbf{V})^T \right)$$

Substituting this into equation 4.4.2, gives the Incompressible Newtonian Navier-Stokes momentum equation,

$$\begin{aligned}
\rho \frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} &= -\nabla P + \nabla \cdot \left[\mu \left(\nabla \mathbf{V} + (\nabla \mathbf{V})^T \right) \right] + \rho \mathbf{f} \\
\rho \frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} &= -\nabla P + \mu \nabla^2 \mathbf{V} + \mu \nabla \cancelto{0}{(\nabla \cdot \mathbf{V})} + \rho \mathbf{f} \\
\rho \frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} &= -\nabla P + \mu \nabla^2 \mathbf{V} + \rho \mathbf{f}
\end{aligned}$$

Assumption 2 - Inviscid Flow

To make our base case of the Bernoulli's equation the most simple, we can make the assumption of negligible viscosity [2], this means that the equation above can be simplified to Euler's equation,

$$\begin{aligned}
\rho \frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} &= -\nabla P + \cancelto{0}{\mu \nabla^2 \mathbf{V}} + \rho \mathbf{f} \\
\rho \frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t} &= -\nabla P + \rho \mathbf{f}
\end{aligned}$$

Assumption 3 - Steady State

Under steady state conditions there is no variation in any of the flow properties with time, so that $\partial / \partial\, t = 0$. This means that we can say that from the equation above by the definition of the substantial derivative,

$$\begin{aligned}
\rho \left(\cancelto{0}{\frac{\partial\, \mathbf{V}}{\partial\, t}} + (\mathbf{V} \cdot \nabla)\, \mathbf{V} \right) &= -\nabla P + \rho \mathbf{f} \\
\rho\, (\mathbf{V} \cdot \nabla)\, \mathbf{V} &= -\nabla P + \rho \mathbf{f}
\end{aligned}$$

Assumption 4 - Irrotational flow

Irrotational flow occurs where the curl (Section A.2.3) of the velocity (vorticity) of the fluid is zero everywhere, by definition $\nabla \times \mathbf{V} = 0$ [3]. This means that the circulation around any arbitrary loop in an irrotational flow pattern is zero. If a flow is inviscid and at any point in time the fluid is irrotational it will always be irrotational. The equation above can have the left-hand-side expanded with a vector identity, thus,

$$\begin{aligned}
\rho \left(\nabla \frac{|\mathbf{V}|^2}{2} - \mathbf{V} \times \cancelto{0,\ \text{irrotational}}{(\nabla \times \mathbf{V})} \right) &= -\nabla P + \rho \mathbf{f} \\
\rho \nabla \frac{|\mathbf{V}|^2}{2} &= -\nabla P + \rho \mathbf{f}
\end{aligned}$$

Assumption 5 - Gravity is the only external force

If the only external force acting on our system is gravity, and the upward direction is defined as the positive z-direction, then it can be defined that $\mathbf{f} = -g\nabla z$, such that inserting into the equation above gives,

$$\rho\nabla\frac{|\mathbf{V}|^2}{2} = -\nabla P - \rho g\nabla z$$
$$\nabla\left(P + \rho\frac{|\mathbf{V}|^2}{2} + \rho g z\right) = 0$$

Assumption 6 - One-Dimensional Flow

To satisfy the fact that the gradient equals zero, the integral must equal a constant, if we only have one-dimensional flow then $\mathbf{V} = u$ (as invicid flow the velocity profile is flat as in Figure 3.7(a)) and $z = h$ (the height). We can then apply our expression over a large distance, e.g. pipe, rather than a small fluid element. This then means that the equation above becomes,

$$P + \frac{\rho u^2}{2} + \rho g h = \text{Constant}$$

Looking at the units of the various terms in Bernoulli's Equation:

$$\frac{P}{\rho g}: \qquad \frac{\text{kg m}^{-1}\,\text{s}^{-2}}{\text{kg m}^{-3}\,\text{m s}^{-2}} = \text{m}$$
$$\frac{u^2}{2g}: \qquad \frac{(\text{m s}^{-1})^2}{\text{m s}^{-2}} = \text{m}$$
$$h: \qquad \text{m}$$

Clearly, the terms must have consistent units, and as it happens, in this form of the equation, the units are in metres. This has led to the use of the term head to describe the different terms. The first term is the pressure head (in fact, we introduced this term earlier, in the context of hydrostatics). The second term is the velocity head, while the third term is the potential head.

4.5 Bernoulli's Equation in terms of Energy

Although Bernoulli's equation can be derived from the momentum conservation equation, we can also think about the terms in terms of energy. The energy must also be conserved in flowing fluids, but as the fluid flows, energy may be converted from one form to another – pressure energy, potential energy, kinetic energy, thermal energy and mechanical work can be interconverted.

Consider fluid flowing within a pipe as illustrated in Figure 4.4, in which the cross sectional area of the pipe changes, so the fluid velocity changes, and the height of the fluid in the pipe also changes, i.e. the fluid is having to be pushed "uphill", for example. The mass flowrate of the fluid is $\dot{M}$, and the fluid is undergoing steady flow. Let us take equation 4.4.3 and multiply this by $\dot{M}/\rho$ (the volumetric flow rate) we get,

$$\underbrace{\frac{\dot{M}P}{\rho}}_{\text{Pressure Energy}} + \underbrace{\frac{\dot{M}u^2}{2}}_{\text{Kinetic Energy}} + \underbrace{\dot{M}gh}_{\text{Potential Energy}} = \text{Constant} \tag{4.5.1}$$

We will consider the energy of the system bounded by the cross sectional planes at points 1 and 2 in Figure 4.4.

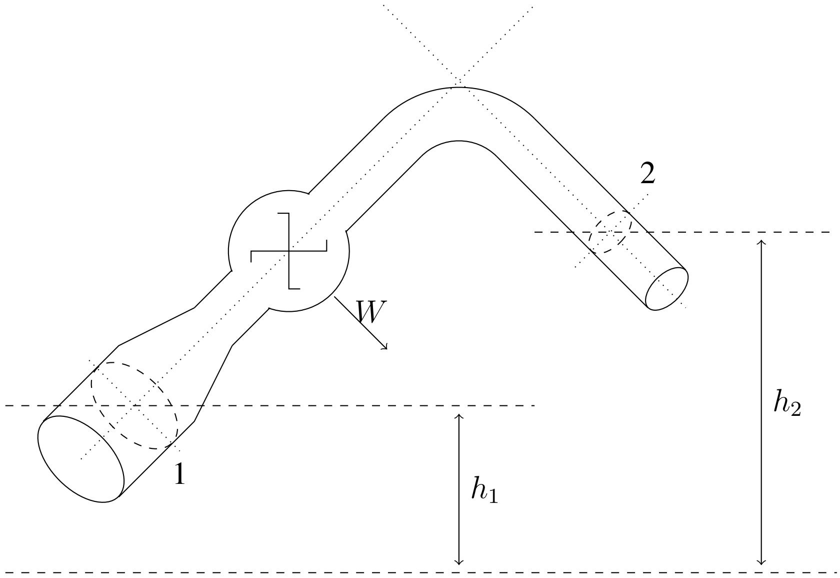


Figure 4.4: Diagram of fluid flow along a pipe, as a basis for performing an energy balance between points 1 and 2.

4.5.1 Potential Energy

At point 1, the potential energy of the fluid is equal to,

$$E_{\text{pot},1} = Mgh_1$$

The rate at which potential energy is added into the system at point 1 is therefore equal to,

$$\dot{E}_{\text{pot},1} = \dot{M}gh_1 \tag{4.5.2}$$

Similarly, the rate at which potential energy leaves the system at point 2 is $\dot{M}gh_2$.

4.5.2 Kinetic Energy

At point 1, the kinetic energy of the fluid is,

$$E_{\text{kin},1} = 0.5Mu_1^2$$

The rate of addition of kinetic energy to the system at point 1 is therefore,

$$\dot{E}_{\text{kin},1} = 0.5\dot{M}u_1^2 \tag{4.5.3}$$

Similarly, the rate at which kinetic energy leaves the system at point 2 is $0.5\dot{M}u_2^2$. The relationship between u_1 and u_2 depends on the cross sectional areas at points 1 and 2 and is given by the mass continuity equation.

4.5.3 Pressure Energy

The rate at which energy is added to the system due to the pressure at point 1, P_1, is found by considering a piston acting on the fluid at that point, as illustrated in Figure 4.5.

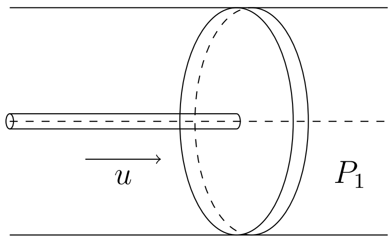


Figure 4.5: Piston acting against a pressure P_1 over a distance $\mathrm{d}\,l$ at a velocity u.

The force exerted by the piston, F, is equal to the pressure times the cross sectional area,

$$F = P_1 A_1$$

Work = force × distance moved in the direction of the force, therefore,

$$\begin{aligned} W &= F\mathrm{d}\,l = P_1 A_1 \mathrm{d}\,l \\ &= \text{pressure} \times \text{volume displaced by the piston} \end{aligned}$$

Rate of energy input = work per unit time, therefore,

$$\dot{E}_{\text{press}} = \frac{\mathrm{d}\,W}{\mathrm{d}\,t} = P_1 A_1 \frac{\mathrm{d}\,l}{\mathrm{d}\,t} = P_1 A_1 u_1 = \frac{P_1 \dot{M}}{\rho_1} \tag{4.5.4}$$

as $u = \mathrm{d}\,l/\mathrm{d}\,t$, the fluid's velocity. So rate of energy input to the system is $P_1 \dot{M}_1/\rho_1$. Similarly, rate of energy done by the system due to the pressure at point 2 is $P_2 \dot{M}_2/\rho_2$.

4.5.4 Other Energies

Work

The fluid may be doing mechanical work at a certain rate, $\dot{W}$, e.g.

- by flowing through a turbine to produce power — in which case $\dot{W}$ would have a positive sign, as work done by the system; or
- there may be a pump imparting energy to the fluid — in which case $\dot{W}$ would have a negative sign, as work done on the system.

Thermal Losses

The fluid may also gain/lose energy as it flows along the pipe, $\dot{Q}$ (not to be confused with the volumetric flowrate of the fluid, Q, as noted before), e.g.

- The fluid would convert some of its energy to thermal energy as it flows along (due to the internal friction of the fluid, i.e. its viscosity), and more so if there are bends and fittings in the pipe. For the fluid to stay at the same temperature (and hence same internal energy) this heat is lost to the surroundings.

- There could also energy inputs to the fluid via heat transfer or chemical reaction.

4.5.5 Extended Bernoulli's Equation

Adding the Work and thermal energies to our Bernoulli's equation, equation 4.4.4, produces,

$$\frac{P_1\dot{M}}{\rho_1} + \frac{\dot{M}u_1^2}{2} + \dot{M}gh_1 = \frac{P_2\dot{M}}{\rho_2} + \frac{\dot{M}u_2^2}{2} + \dot{M}gh_2 + \dot{W} + \dot{Q} \tag{4.5.5}$$

Now, the mass flowrate, $\dot{M}$, is constant, so can be divided through. It is also conventional to divide through by the acceleration due to gravity (as we did above),

$$\frac{P_1}{\rho_1 g} + \frac{u_1^2}{2g} + h_1 = \frac{P_2}{\rho_2 g} + \frac{u_2^2}{2g} + h_2 + \frac{\dot{W}}{\dot{M}g} + \frac{\dot{Q}}{\dot{M}g} \tag{4.5.6}$$

If no work is done on or by the system, then the second to last term disappears. If no heat is transferred to/from the system and there is no chemical reaction, then the last term is only due to the losses due to viscosity which are generally described by the term Δh_f. Therefore the equation becomes,

$$\frac{P_1}{\rho_1 g} + \frac{u_1^2}{2g} + h_1 = \frac{P_2}{\rho_2 g} + \frac{u_2^2}{2g} + h_2 + \Delta h_f \tag{4.5.7}$$

We will extend Bernoulli's Equation to real fluids later with losses and learn how to calculate the Δh_f term for them in Chapter 6.

4.6 References

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Chapter 5

Applications of Bernoulli's Equation

Contents

5.1 Chapter 5 ILOs

ILO 5.1. Analyse the flow from tanks using the Bernoulli's equation.

ILO 5.2. Apply Bernoulli's equation to calculation in flow measuremnent devices.

5.2 Introduction

In Chapter 4 we developed Bernoulli's equation, equation 4.4.4 or equation 4.5.7 including frictional effects. In this chapter, we will explore how we can use Bernoulli's equation to look at real systems and calcuate the key fluid parameters.

5.3 Flow from a Tank

5.3.1 Tanks with Constant Height

Consider a tank of cross sectional area A_1 containing a liquid to a height h_1, which flows from the bottom of the tank through a small hole of area A_2, as shown in Figure 5.1. The height of the liquid is kept constant by adding liquid at an appropriate rate. Find an expression for the rate of discharge of liquid.

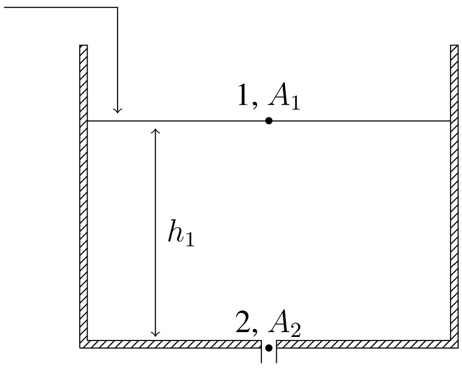

Figure 5.1: Liquid discharging through a hole in the base of a tank.

Apply Bernoulli's Equation between points 1 and 2.

$$\frac{P_1}{\rho_1 g} + \frac{u_1^2}{2g} + h_1 = \frac{P_2}{\rho_2 g} + \frac{u_2^2}{2g} + h_2 + \Delta h_f \tag{5.3.1}$$

Assumptions:

- Ideal (frictionless) flow; $\Delta h_f = 0$
- $u_1 = 0$ (because of the large area of A_1 compared with A_2, the velocity at point 1 is negligible compared with the velocity at point 2)
- $h_2 = 0$ (datum point is chosen at h_2).
- $P_1 = P_2 = P_{\text{atm}}$. Both the top surface and the liquid discharging through the hole are at atmospheric pressure.

Therefore,

$$\begin{aligned} \cancelto{\frac{P_{\text{atm}}}{\rho_1 g}}{\frac{P_1}{\rho_1 g}} + \cancelto{0}{\frac{u_1^2}{2g}} + h_1 &= \cancelto{\frac{P_{\text{atm}}}{\rho_1 g}}{\frac{P_2}{\rho_2 g}} + \frac{u_2^2}{2g} + \cancelto{0}{h_2} + \cancelto{0}{\Delta h_f} \\ h_1 &= \frac{u_2^2}{2g} \\ u_2 &= \sqrt{2gh_1} \end{aligned} \tag{5.3.2}$$

The volumetric flowrate of fluid through the hole is then given by,

$$Q = u_2 A_2 = A_2 \sqrt{2gh_1} \tag{5.3.3}$$

and the mass flowrate by,

$$\dot{M} = \rho u_2 A_2 = \rho A_2 \sqrt{2gh_1} \tag{5.3.4}$$

In practice, friction effects arising from the fluid's viscosity will not be negligible, such that $\Delta h_f \neq 0$ and the velocity of the fluid discharging from the hole will be smaller. This is allowed for in practice by including an empirically (i.e. experimentally) determined discharge coefficient, C_D, where $C_D < 1$,

Discharge from a tank with constant level

$$u = C_D \sqrt{2gh_1} \tag{5.3.5}$$

5.3.2 Tank with Variable Height

In the situation in which the tank is emptying and the liquid discharging through the hole is not replaced, such that h_1 is no longer constant, the varying flowrate can be described, and the time to empty the tank determined. Note that this is now an unsteady state situation, compared with the steady state system described above. We would need to assume that the velocity of the exiting liquid has the same relation to the liquid's height in the unsteady situation as it did in the steady situation, and check our assumption by experiment. Even so, such an analysis indicates the broad relationships between the factors determining the time to empty the tank, and gives a basis for an initial estimate of that time.

Mass balance,

$$\begin{array}{ccccc} \text{Mass flow in} & - & \text{Mass flow out} & = & \text{Rate of change of mass in tank} \\ 0 & - & \rho u_2 A_2 & = & \dfrac{\mathrm{d}\,(\rho A_1 h_1)}{\mathrm{d}\,t} \end{array}$$

Based on our previous analysis for the steady state situation, assuming that the discharge velocity has the same relation to h even though h is now changing from h_1 initially to h_2 finally, we can say,

$$u_2 = C_D \sqrt{2gh}$$

Substituting this expression for u_2 gives,

$$\begin{aligned} A_1 \frac{\mathrm{d}\,h}{\mathrm{d}\,t} &= -\,A_2 C_D \sqrt{2gh} \\ \frac{\mathrm{d}\,h}{\sqrt{h}} &= -\,\frac{A_2 C_D \sqrt{2g}}{A_1} \mathrm{d}\,t \end{aligned}$$

For the tank to empty, h varies from h_1 initially (at time $t = 0$) to h_2 finally (at time $t = t$). So, we can integrate both sides of the equation:

$$\begin{aligned} \int_{h_1}^{h_2} \frac{\mathrm{d}\,h}{\sqrt{h}} &= -\,\frac{A_2 C_D \sqrt{2g}}{A_1} \int_0^t \mathrm{d}\,t \\ \left[2\sqrt{h}\right]_{h_1}^{h_2} &= -\,\frac{A_2 C_D \sqrt{2g}}{A_1} t \\ t &= \frac{2A_1}{A_2 C_D \sqrt{2g}} \left(\sqrt{h_1} - \sqrt{h_2}\right) \end{aligned}$$

If $h_2 = 0$, then,

$$t = \frac{2A_1}{A_2 C_D \sqrt{2g}} \sqrt{h_1} \tag{5.3.6}$$

In other words, the time to empty the tank varies with the square root of the initial height of the liquid in the tank. Doubling the amount of liquid in the tank initially will lead to a 41% increase in the time to empty the tank (as $\sqrt{2} \approx 1.414$).

5.4 Flow Measurement

Bernoulli's equation can be applied to address a critical issue of interest to chemical and petroleum engineers – that of measuring the flowrate of fluid in a pipe. The principle is that the flow of fluid is either accelerated or retarded (i.e. change in velocity), and the resulting change in pressure measured (as kinetic and pressure energy interconvert), with Bernoulli's equation then used to interpret the change and relate it to the fluid flowrate. Devices that employ this principle include:

The Pitot tube, in which a small element of fluid is brought to rest, such that its kinetic energy is converted into pressure energy – by measuring the increase in the pressure, the original kinetic energy and hence the velocity can be determined

The Orifice meter, in which fluid passes through a sudden constriction, increasing its velocity and hence its kinetic energy, which is then related to the reduction in pressure

The Venturi meter, in which the fluid passes through a gradual constriction to increase its velocity and reduce its pressure.

5.4.1 Velocity Profile Measurement – The Pitot Tube

The principle of the Pitot tube was first used by Henri Pitot in 1732 for measuring velocities of water in the River Seine [3]. Figure 5.2 shows a diagram of a modern Pitot tube, which consists of two concentric tubes arranged to join the two arms of a manometer. Fluid is flowing at point 1 with a velocity u_1. At point 2, the presence of the nose of the Pitot tube causes a stagnation point, at which the velocity is zero, because fluid is unable to flow into the inner tube. Thus the pressure at point 2 is higher than at point 1, as the kinetic energy has been converted into pressure. At point 3, the fluid is flowing with (close to) its original velocity, so its pressure is lower than the pressure at point 2.

The position of the hole at point 3 is critical (in fact, several holes are drilled around the circumference of the outer tube at that point). The presence of the head of the Pitot tube disrupts the flow slightly and affects the measured pressure, as does the presence of the stem. On a standard Pitot tube, the distance from the head to the stem is 14 tube diameters, and the holes are located 6 diameters from the head and 8 diameters from the stem – at this point, the effects on the measured pressure are equal and opposite and cancel each other out. A well-designed and correctly operated Pitot tube is accurate to within about 1%.

In order to relate the measured height difference in the manometer to the velocity, we

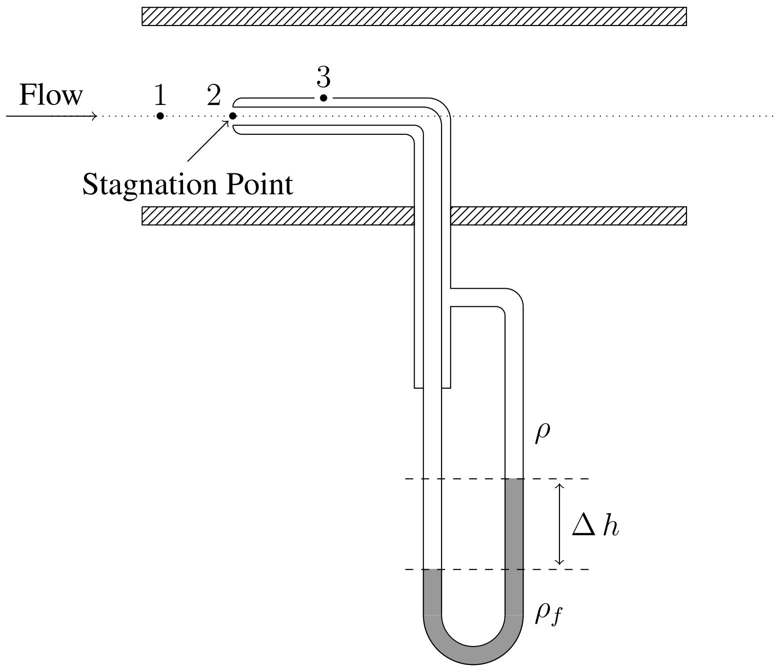

Figure 5.2: Pitot tube, used for measuring the local flow velocity in a fluid.

apply Bernoulli's Equation between points 2 and 3:

$$\frac{P_2}{\rho_2 g} + \frac{u_2^2}{2g} + h_2 = \frac{P_3}{\rho_3 g} + \frac{u_3^2}{2g} + h_3 + \Delta h_f$$

Assumptions:

- Ideal (frictionless) flow; $\Delta h_f = 0$.
- $u_2 = 0$ (fluid is brought to a stagnation point).
- $h_2 = h_3$ (the Pitot tube is horizontal).
- $u_3 = u$ (the general fluid velocity).
- $\rho_2 = \rho_3 = \rho$ (incompressible fluid).

Therefore,

$$\frac{P_2}{\rho g} + \cancelto{0}{\frac{u_2^2}{2g}} + \cancelto{0}{h_2} = \frac{P_3}{\rho g} + \frac{u^2}{2g} + \cancelto{0}{h_3} + \cancelto{0}{\Delta h_f}$$

$$\frac{P_2}{\rho g} = \frac{P_3}{\rho g} + \frac{u^2}{2g}$$

$$P_2 - P_3 = \frac{\rho u^2}{2} \tag{5.4.1}$$

The pressure difference is indicated by the hydrostatic height difference in the manometer (where ρ_F is the density of the manometer fluid). This is found as in our manometer relation (equation 2.6.3), such that,

$$P_2 - P_3 = (\rho_F - \rho)\, g \Delta h = \frac{\rho u^2}{2} \tag{5.4.2}$$

Therefore, the velocity is given by:

Pitot Tube

$$u = \sqrt{\frac{2\,(\rho_F - \rho)\,g\Delta\,h}{\rho}} \tag{5.4.3}$$

Note that this is the local velocity of the fluid at the particular location in the pipe (not the average velocity across the whole pipe). By moving the Pitot tube across a pipe (or river, or other fluid flow system of interest), one can measure the velocity profile.

5.4.2 The Orifice Meter

In contrast with the Pitot tube, which slows the flow and measures the resulting increase in pressure, orifice and Venturi meters cause the fluid to flow through a restriction in order to increase the flow velocity, and measure the resulting decrease in pressure [2]. Also in contrast with the Pitot tube, which measures the local velocity at a particular point within the flow, these two meters give a measure of the average flow velocity across the whole cross sectional area of the pipe.

Figure 5.3 illustrates the orifice meter. An orifice plate is inserted across the cross section of the flow. The hole in the orifice plate is smaller than the pipe diameter, and (according to continuity) the flow velocity must increase in order to get the fluid through the constriction.

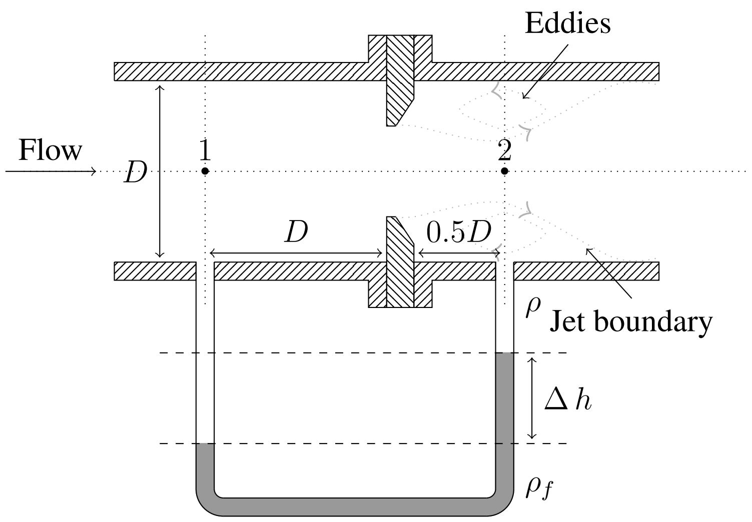

Figure 5.3: Orifice meter, in which an orifice plate inserted into the flow increases the fluid velocity, causing a decrease in pressure, which is measured with a manometer.

Upstream of the orifice, the flow cannot suddenly turn the corner when it meets the orifice plate. Instead, the streamlines converge, to form a cone of streamlines as they approach the orifice. Again, these streamlines cannot suddenly change direction as they pass through the orifice, so they continue to converge for some distance past the orifice, to form what is called a vena contracta, a jet of fast moving fluid surrounded by slowly moving eddies near the wall. As this jet mixes with the slower moving fluid downstream, it slows down and the streamlines diverge to fill the entire pipe cross section once again. The flow downstream of the orifice is highly turbulent, such that most of the kinetic energy of the jet is dissipated as heat and is not recoverable as pressure energy. Thus the pressure drop over an orifice meter is high (which may not be acceptable in some circumstances).

The orifice plate has a sharp edge on its upstream side, rather than a right angle, so that the flow through the plate itself is not affected by the walls of the plate. Similarly, to avoid flow effects from upstream bends and fittings, the orifice plate should be located at least 30 pipe diameters (and preferably at least $50D$) downstream of any such fittings.

The jet velocity is at its fastest when the vena contracta is at its narrowest, which occurs about half a pipe diameter downstream of the orifice. The pressure at this point will therefore be lowest, and this is where one arm of the manometer is located, with the other arm placed one pipe diameter upstream of the plate (The velocity profile across the jet is also constant at this point.). Labelling the upstream point as point 1, and the minimum of the vena contracta as point 2, and noting that $h_1 = h_2$, applying Bernoulli's Equation gives,

$$\frac{P_1}{\rho_1 g} + \frac{u_1^2}{2g} + h_1 = \frac{P_2}{\rho_2 g} + \frac{u_2^2}{2g} + h_2 + \Delta h_f \tag{5.4.4}$$

Assumptions:

- Ideal (frictionless) flow; $\Delta h_f = 0$.
- $h_1 = h_2$ (the orifice is horizontal).
- $\rho_1 = \rho_2 = \rho$ (incompressible fluid).

Therefore,

$$\begin{aligned}
\frac{P_1}{\rho g} + \frac{u_1^2}{2g} + \cancelto{0}{h_1} &= \frac{P_2}{\rho g} + \frac{u_2^2}{2g} + \cancelto{0}{h_3} + \cancelto{0}{\Delta h_f} \\
\frac{P_1}{\rho g} + \frac{u_1^2}{2g} &= \frac{P_2}{\rho g} + \frac{u_2^2}{2g} \\
P_1 + \frac{\rho u_1^2}{2} &= P_2 + \frac{\rho u_2^2}{2} \\
P_1 - P_2 &= \frac{\rho}{2}\left(u_2^2 - u_1^2\right)
\end{aligned} \tag{5.4.5}$$

As before for the Pitot tube, the pressure difference is related to the height difference in the manometer (equation 2.6.3),

$$P_1 - P_2 = (\rho_F - \rho)\, g \Delta h = \frac{\rho}{2}\left(u_2^2 - u_1^2\right) \tag{5.4.6}$$

By mass continuity,

$$u_1 = u_2 \frac{A_2}{A_1} \tag{5.4.7}$$

Thus,

$$(\rho_F - \rho)\, g \Delta h = \frac{\rho}{2} u_2^2 \left(1 - \left(\frac{A_2}{A_1}\right)^2\right) \tag{5.4.8}$$

Therefore,

$$u_2 = \sqrt{\frac{2(\rho_F - \rho)\, g \Delta h}{\rho\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}} \tag{5.4.9}$$

The flow rate is therefore,

$$Q = u_2 A_2 = A_2 \sqrt{\frac{2(\rho_F - \rho)\, g\Delta h}{\rho\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}} \tag{5.4.10}$$

Now, the problem is we don't know the cross-sectional area of the vena contracta, A_2. We do know, of course, the cross-sectional area of the orifice itself, which we'll call A_o. We can define a coefficient of contraction, C_c, which is the ratio of the cross-sectional areas of the vena contracta and the orifice: $A_2 = C_c A_o$.

In addition, frictional effects due to viscosity cause the velocity in the vena contracta to be less that the velocity predicted by Bernoulli's equation. A coefficient of velocity, C_v, can be defined as the ratio of the actual mean velocity to the ideal velocity in the vena contracta.

In combination, these two effects – the contraction of the area of the jet and the slowing effects of friction – mean that the discharge (the volumetric flowrate through the orifice) is less than one might expect based on Bernoulli's equation for a given area of the orifice. Combining the two effects, we can define a coefficient of discharge, C_D, as the ratio of the actual discharge to the ideal discharge, such that C_D includes the effects of C_c and C_v. It also includes the necessary conversions from A_2 to A_o and any calibration factors, to give:

Orifice Meter

$$Q_{\text{actual}} = C_D A_o \sqrt{\frac{2(\rho_F - \rho)\, g\Delta h}{\rho\left(1 - \left(\frac{A_o}{A_1}\right)^2\right)}} = C_D A_o \sqrt{\frac{2(P_1 - P_2)}{\rho\left(1 - \left(\frac{A_o}{A_1}\right)^2\right)}} \tag{5.4.11}$$

The value of the discharge coefficient depends on the specific design of the orifice plate (e.g. the ratio of orifice diameter to pipe diameter, and the precise positioning of the pressure tappings) and on the value of the Reynolds Number in the orifice (where the diameter D used in the calculation of the Reynolds Number is the orifice diameter). For $\text{Re} > 10^4$, C_D typically has a value of about 0.61. For $\text{Re} < 10^4$, the relationship is complex, rising with increasing Re to a maximum, and then decreasing towards a steady value of 0.61.

Orifice plates have the advantages that they are cheap and take up little space. They have the disadvantage that they incur a large permanent pressure drop.

5.4.3 The Venturi Meter

In contrast to an orifice meter, which uses an orifice plate to give a sudden contraction through which the fluid must flow, a Venturi meter uses a gradual contraction followed by a gradual expansion [1]. Figure 5.4 illustrates a Venturi meter, which consists of a converging section of pipe, leading to a short parallel-sided throat, followed by a diverging section known as a diffuser.

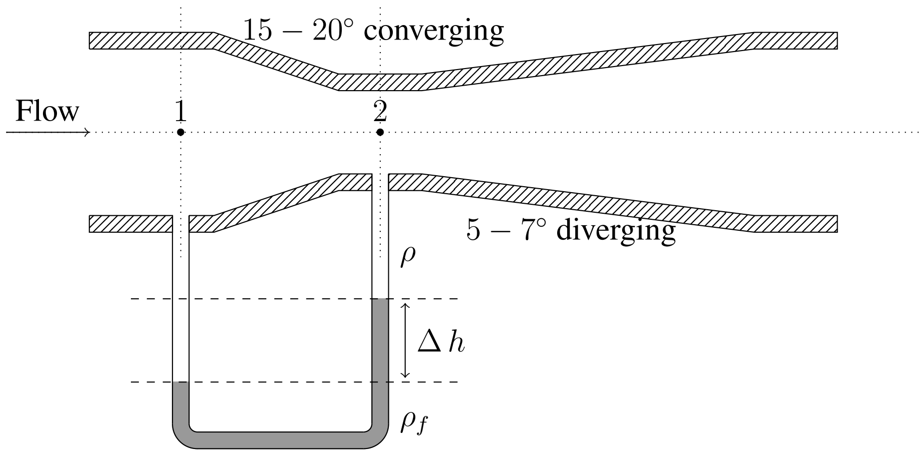

Figure 5.4: Venturi meter, in which a converging section of pipe increases the fluid velocity, causing a decrease in pressure, which is measured with a manometer. The gently diverging section then recovers most of the increased velocity head as pressure.

As for the orifice meter, the constriction causes the fluid velocity to increase, and the pressure difference is measured with a manometer. However, the gradually converging section means that there is no vena contracta, so the coefficient of contraction, C_c, is unity. Similarly, the gradually diverging section avoids high turbulence and the formation of eddies, so recovers most of the kinetic energy as pressure. Thus Venturi meters incur only a small pressure drop, which is attractive when there is only a small pressure drop driving the flow in the first place. They are also more accurate than an orifice meters. However, they take up more space than an orifice plate and are more expensive to construct. The coefficient of discharge, C_D, of a Venturi meter is typically around $0.96 - 0.98$ for fluids of low viscosity. As with the orifice meter, they must be installed in a straight piece of pipe with no bends or fittings at least 30 pipe diameters upstream and 5 pipe diameters downstream of the meter.

The angle of divergence in the diffuser determines the extent of pressure recovery, low angles giving more recovery, but requiring a longer meter. In practice an angle of $5 - 7°$ gives a good compromise. Larger angles are sometimes used, to reduce the length and cost of the meter, but at the cost of a larger pressure drop.

The ratio of throat diameter to pipe diameter is typically 0.5, giving a ratio of areas of 0.25, and a ratio of velocities of 4. A smaller ratio gives a greater fluid velocity in the throat and hence a greater and more accurately measured pressure difference. However, the subsequent dissipation of kinetic energy as heat in the diffuser is also greater (less pressure recovery), while very low pressures in the throat can cause dissolved gases to come out of solution or vaporisation to occur, both of which would affect the measurement.

Applying Bernoulli's equation across a Venturi meter gives the same result as for an orifice plate, but with a different value of the discharge coefficient, and with A_2, the throat diameter, replacing the orifice diameter:

Venturi Meter

$$Q_{\text{actual}} = C_D A_2 \sqrt{\frac{2\left(\rho_F - \rho\right) g \Delta h}{\rho\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}} = C_D A_2 \sqrt{\frac{2\left(P_1 - P_2\right)}{\rho\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}} \tag{5.4.12}$$

Table 5.1 summarises the differences between orifice and Venturi meters.

A flow nozzle is another design of flow meter, which uses a curved entrance to an orifice in order to accelerate the fluid, but does not have a diverging zone to recover the velocity head. It has a high coefficient of discharge (~ 0.99), but a high pressure loss. It is cheaper than a Venturi meter.

Table 5.1: Comparison of Orifice and Venturi meter.

Comparison	Orifice Plate	Venturimeter
Cost	Cheaper	More Expensive
Adaptability (change area ratio)	Versatile	Fixed area ratio
Precision	Less accurate	More accurate
Permanent pressure loss	High	Low
C_D (typical)	$0.61 - 0.65$	$0.96 - 0.98$

5.5 References

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Chapter 6

Friction Losses - Ideal Fluids

Contents

6.1 Chapter 6 ILOs

ILO 6.1. Identify the general integral force balance as derived from the general Navier-Stokes equation.

ILO 6.2. Calculate frictional losses in pipe flow and apply to real systems.

ILO 6.3. Compute losses in pipelines for different pipe fittings.

6.2 Force Balance on Fluid Element

The general form of the Navier-Stokes equation (Cauchy momentum equation), equation 4.4.2, can be defined over a control volume as shown in Figure 6.1. In this case, we have knowledge of the boundaries, this means that the surface integrals (over Ω) are more convenient for us to evaluate, so we have the general force balance,

$$\underbrace{\frac{\partial}{\partial t}\oiiint_{\Omega}\rho\mathbf{V}\mathrm{d}\,\Omega}_{\text{Total Force, }\mathbf{F}=\frac{\partial m\mathbf{V}}{\partial t}} = +-\underbrace{\oiint_{\Gamma}\rho\mathbf{V}\cdot(\mathbf{V}\cdot\mathbf{n})\,\mathrm{d}\,\Gamma}_{\text{Momentum Force}} +-\underbrace{\oiint_{\Gamma}P\mathbf{n}\mathrm{d}\,\Gamma}_{\text{Pressure Force}} +\underbrace{\oiint_{\Gamma}\boldsymbol{\tau}\cdot\mathbf{n}\mathrm{d}\,\Gamma}_{\text{Shear Force}} +-\underbrace{\oiint_{\Gamma}\rho g z\cdot\mathbf{n}\mathrm{d}\,\Gamma}_{\text{Gravity Force}} \tag{6.2.1}$$

The full derivation can be seen in Derivation 6.1.

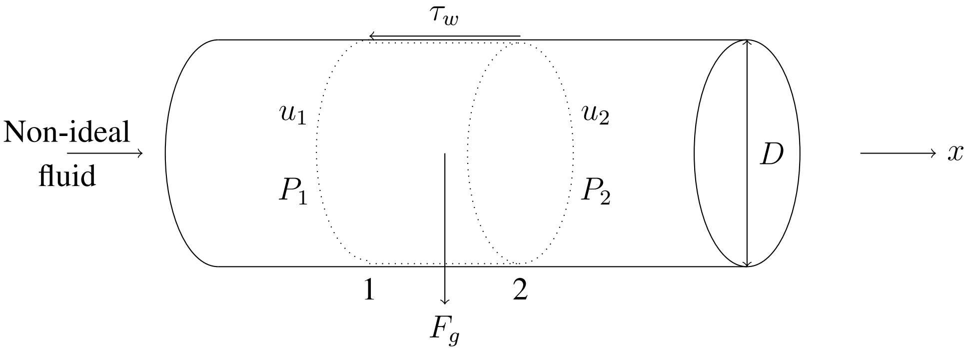

Figure 6.1: Force balance over a control volume.

Derivation 6.1: Force Balance.

Taking the general form of the Navier-Stokes equation (Cauchy momentum equation), equation 4.4.2, we can write this as an integral over the general volume Ω,

$$\oiiint_{\Omega}\rho\frac{\mathrm{D}\,\mathbf{V}}{\mathrm{D}\,t}\mathrm{d}\,\Omega = -\oiiint_{\Omega}\nabla P\mathrm{d}\,\Omega + \oiiint_{\Omega}\nabla\cdot\boldsymbol{\tau}\mathrm{d}\,\Omega + \oiiint_{\Omega}\rho\mathbf{f}\mathrm{d}\,\Omega$$

Expanding the substantial derivative gives,

$$\oiiint_{\Omega}\rho\left(\frac{\partial\,\mathbf{V}}{\partial\,t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\right)\mathrm{d}\,\Omega = -\oiiint_{\Omega}\nabla P\mathrm{d}\,\Omega + \oiiint_{\Omega}\nabla\cdot\boldsymbol{\tau}\mathrm{d}\,\Omega + \oiiint_{\Omega}\rho\mathbf{f}\mathrm{d}\,\Omega$$

The momentum part of the integral can then be separated and moved to the right-hand side as,

$$\oiiint_{\Omega}\rho\frac{\partial\,\mathbf{V}}{\partial\,t}\mathrm{d}\,\Omega = -\oiiint_{\Omega}\rho\,(\mathbf{V}\cdot\nabla)\,\mathbf{V}\mathrm{d}\,\Omega - \oiiint_{\Omega}\nabla P\mathrm{d}\,\Omega + \oiiint_{\Omega}\nabla\cdot\boldsymbol{\tau}\mathrm{d}\,\Omega + \oiiint_{\Omega}\rho\mathbf{f}\mathrm{d}\,\Omega$$

If we want to apply this equation over a control volume as shown in Figure 6.1, then we have knowledge of the boundaries (also assuming that only external force acting on our system is gravity, and the upward direction is defined as the positive z-direction, then $\mathbf{f} = -g\nabla z$ as we did in Chapter 4). This means that surface integrals are more convenient for us to evaluate as we did for equation 4.3.4. Therefore applying the divergence theorem to the right-hand-side terms[a],

$$\oiiint_{\Omega} \rho \frac{\partial \mathbf{V}}{\partial t} \mathrm{d}\,\Omega = -\oiint_{\Gamma} \rho \mathbf{V} \cdot (\mathbf{V} \cdot \mathbf{n})\, \mathrm{d}\,\Gamma - \oiint_{\Gamma} P \mathbf{n} \mathrm{d}\,\Gamma + \oiint_{\Gamma} \boldsymbol{\tau} \cdot \mathbf{n} \mathrm{d}\,\Gamma - \oiint_{\Gamma} \rho g z \cdot \mathbf{n} \mathrm{d}\,\Gamma$$

Also applying the Reynolds transport theorem to the left-hand-side gives,

$$\frac{\partial}{\partial t} \oiiint_{\Omega} \rho \mathbf{V} \mathrm{d}\,\Omega = -\oiint_{\Gamma} \rho \mathbf{V} \cdot (\mathbf{V} \cdot \mathbf{n})\, \mathrm{d}\,\Gamma - \oiint_{\Gamma} P \mathbf{n} \mathrm{d}\,\Gamma + \oiint_{\Gamma} \boldsymbol{\tau} \cdot \mathbf{n} \mathrm{d}\,\Gamma - \oiint_{\Gamma} \rho g z \cdot \mathbf{n} \mathrm{d}\,\Gamma$$

[a]noting that $\nabla P = \nabla \cdot \mathbf{I} P$

6.3 Application to a Short Pipe Section

We can now expand the force balance, equation 6.2.1, over our control volume in the pipe. In this case we only have one direction x. In this case the pipe element is not moving and at steady-state thus,

$$F_{T,x} = \frac{\partial}{\partial t} \oiiint_{\Omega} \rho \mathbf{V} \mathrm{d}\,\Omega = 0 \tag{6.3.1}$$

The momentum force applies over the cross-section area at point 1 and 2, but not on the wall, as the velocity in this case is parallel to the walls, thus,

$$\begin{aligned} F_{m,x} &= -\oiint_{\Gamma} \rho \mathbf{V} \cdot (\mathbf{V} \cdot \mathbf{n})\, \mathrm{d}\,\Gamma \\ &= -\left(\rho A_1 u_1 (-u_{1,x}) + \rho A_2 u_2 (u_{2,x})\right) \\ &= \rho A_1 u_1 u_{1,x} - \rho A_2 u_2 u_{2,x} \end{aligned} \tag{6.3.2}$$

However, in this case $A_1 = A_2$ and thus from the continuity equation we know that $\rho A_1 u_1 = \rho A_2 u_2$, so $u_1 = u_2$, which means that,

$$F_{m,x} = 0 \tag{6.3.3}$$

The pressure force applies over the cross-section area at points 1 and 2, but not on the wall (more accurately is the same in every direction into the wall and thus cancels out), thus,

$$\begin{aligned} F_{p,x} &= -\oiint_{\Gamma} P \mathbf{n} \mathrm{d}\,\Gamma \\ &= -\left(A_1 P_1 (-1) + A_2 P_2 (1)\right) \\ &= A_1 P_1 - A_2 P_2 \\ &= (P_1 - P_2) \frac{\pi}{4} D^2 \end{aligned} \tag{6.3.4}$$

The shear force is applied on the wall of the pipe, but not the cross-sectional areas at points 1 and 2. The shear stress on the wall can be defined as the wall shear stress τ_w, so for the fluid element,

$$F_{s,x} = \oiint_{\Gamma} \boldsymbol{\tau} \cdot \mathbf{n} \mathrm{d}\,\Gamma = -\tau_w A_w = -\tau_w \pi D L \tag{6.3.5}$$

The force due to gravity, F_{gx}, is 0 as the pipe is horizontal.

This means that our force balance gives,

$$
\begin{aligned}
\cancelto{0}{F_{T,x}} &= \cancelto{0}{F_{m,x}} + \cancelto{(P_1 - P_2)(\pi/4)D^2}{F_{p,x}} + \cancelto{-\tau_w \pi D L}{F_{s,x}} + \cancelto{0}{F_{g,x}} \\
0 &= (P_1 - P_2)\frac{\pi}{4}D^2 - \tau_w \pi d L \\
P_1 - P_2 &= \tau_w \frac{4L}{D} \qquad (6.3.6)
\end{aligned}
$$

From the extended form of Bernoulli's equation, equation 4.5.7, with no energy input,

$$\frac{P_1}{\rho_1 g} + \cancelto{0}{\frac{u_1^2}{2g}} + \cancelto{0}{h_1} = \frac{P_2}{\rho_2 g} + \cancelto{0}{\frac{u_2^2}{2g}} + \cancelto{0}{h_2} + \Delta h_f \qquad (6.3.7)$$

Which means that,

$$\frac{P_1 - P_2}{\rho g} = \Delta h_f \qquad (6.3.8)$$

Substituting in for $P_1 - P_2$ from equation 6.3.6 gives,

$$\Delta h_f = \frac{\tau_w}{\rho g}\frac{4L}{D} \qquad (6.3.9)$$

As the friction loss is related to the kinetic energy of the fluid, i.e. the fluid velocity, it is typical to include the kinetic energy term (or velocity head, $u^2/2g$),

$$
\begin{aligned}
\Delta h_f &= \frac{\tau_w}{\rho g}\frac{4L}{D} \cdot \frac{u^2}{2g} \cdot \frac{2g}{u^2} \\
&= \frac{\tau_w}{0.5\rho u^2}\frac{4L}{D}\frac{u^2}{2g} \qquad (6.3.10)
\end{aligned}
$$

The $\tau_w/0.5\rho u^2$ term is often referred to as the friction factor, f, so,

Major Friction Losses

$$\Delta h_f = f\frac{4L}{D}\frac{u^2}{2g} \qquad (6.3.11)$$

known as the Darcy equation, with the fanning friction factor, f.

6.4 Fanning Friction Factor

The fanning friction factor has a complex relationship which depends on the Reynold's number and the dimensionless surface roughness, Figure 6.2. The dimensionless surface roughness is given by the surface roughness, ε, divided by the pipe diameter,

$$\text{Dimensionless surface roughness} = \frac{\varepsilon}{D} \qquad (6.4.1)$$

As with the flow velocity profiles the friction factor behaviour depends on the flow regime.

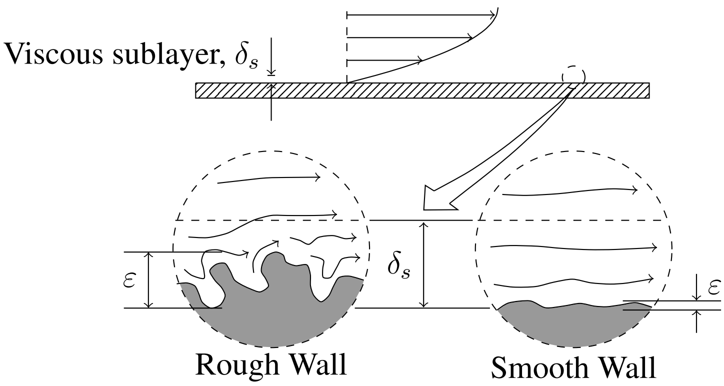

Figure 6.2: Diagram of the surface roughness, ε.

6.4.1 Laminar Flow

In the laminar regime, there is actually an analytically derived solution from first principles independent of the pipe roughness which can be seen by comparison of equation 6.3.11 with equation 3.5.7, this is,

Laminar Friction Factor

$$f = \frac{16}{\text{Re}} \tag{6.4.2}$$

6.4.2 Turbulent Flow

For turbulent flow, it is not easy to determine the functional dependence of the friction factor on the Reynolds number and relative roughness (ε/D). Blasius was the first to derive a law relating to so-called turbulent smooth pipe flows between $3 \times 10^3 < \text{Re} < 2^5$ [2],

$$f = 0.0791\text{Re}^{-0.25} \tag{6.4.3}$$

Much of the information for rough pipes is a result of experiments conducted by Nikuradse in 1933 [8] (and then added to later by others). Three key zones were identified, the first is a critical zone where there is some intermediate behaviour between the laminar flow and the turbulent flow, which is complex and not always repeatable. The second zone is a transition zone were the friction factor depends on both the relative roughness and the Reynolds number. The third zone is the complete turbulence zone where the friction factor is independent of the Reynolds number and only affected by the relative roughness of the pipe, as in [8],

$$f = \frac{1}{\left(2.276 - 4\log\left(\frac{\varepsilon}{D}\right)\right)^2} \tag{6.4.4}$$

These three zones may be physically interpreted, in the first zone the thickness of the laminar boundary layer, δ_s, which is known to decrease with an increasing Reynolds number, is still larger than the average projection, i.e. $\delta_s > \varepsilon$. Therefore, energy losses due to roughness are no greater than those for the smooth pipe. In the transition zone, the thickness of the boundary layer is of the same magnitude as the average projection, i.e. $\delta_s \sim \varepsilon$. Therefore, individual projections extend through the boundary layer and cause vortices which produce an additional loss of energy. As the Reynolds number increases, an increasing number of projections pass through the laminar boundary layer because of the reduction in its thickness. The additional energy loss then changes with Reynolds number. Finally, in the complete turbulence zone the thickness of the boundary layer has become so small that all projections extend through it. The energy loss due to the vortices has now attained a constant value and an increase in the Reynolds number no longer changes the resistance.

Colebrook correlated the original data of Nikuradse in terms of the relative roughness of commercially available pipe materials [3] and generated the expression of,

Turbulant Friction Factor (Colebrook)

$$\frac{1}{\sqrt{f}} = -4\log\left(\frac{\varepsilon/D}{3.7} + \frac{1.256}{\mathrm{Re}\sqrt{f}}\right) \tag{6.4.5}$$

Note that even for smooth pipes the friction factor is not zero. That is, there is a head loss in any pipe, no matter how smooth the surface is made. This is a result of the no-slip boundary condition that requires any fluid to stick to any solid surface it flows over. There is always some microscopic surface roughness that produces the no-slip behavior (and thus $f \neq 0$) on the molecular level, even when the roughness is considerably less than the viscous sublayer thickness.

The Colebrook equation is implicit in f, and determination of friction factor requires tedious iteration. Therefore some solutions have been created; the fist and traditionally used is from Moody who plotted the friction factor against Reynolds number as Figure 6.3 [7].

The other option has been suggested by many researchers and is to reform the equation as an explicit approximation of the Colebrook equation [9, 10]. One of the commonly used versions of this is the Haaland equation [6],

$$\frac{1}{\sqrt{f}} = -3.6\log\left(\left(\frac{\varepsilon/D}{3.7}\right)^{1.11} + \frac{6.9}{\mathrm{Re}}\right) \tag{6.4.6}$$

Table 6.1 shows some typical values of the surface roughness, a more complete list is given in Section B.3.

Table 6.1: Some typical roughness values.

Material	ε / mm
Drawn tubing	0.0015
Commercial steel	0.045
Galvanized iron	0.15
Cast iron	0.25
Concrete	0.3-3.0

Friction Factor

In this course the friction factor used is the fanning friction factor, be careful not to confuse this with the Darcy-Weisbach friction factor which is four times larger

$$f_{\text{Darcy-Weisbach}} = 4f_{\text{fanning}}$$

The Darcy-Weisbach friction factor is often given as λ but this is not always true.

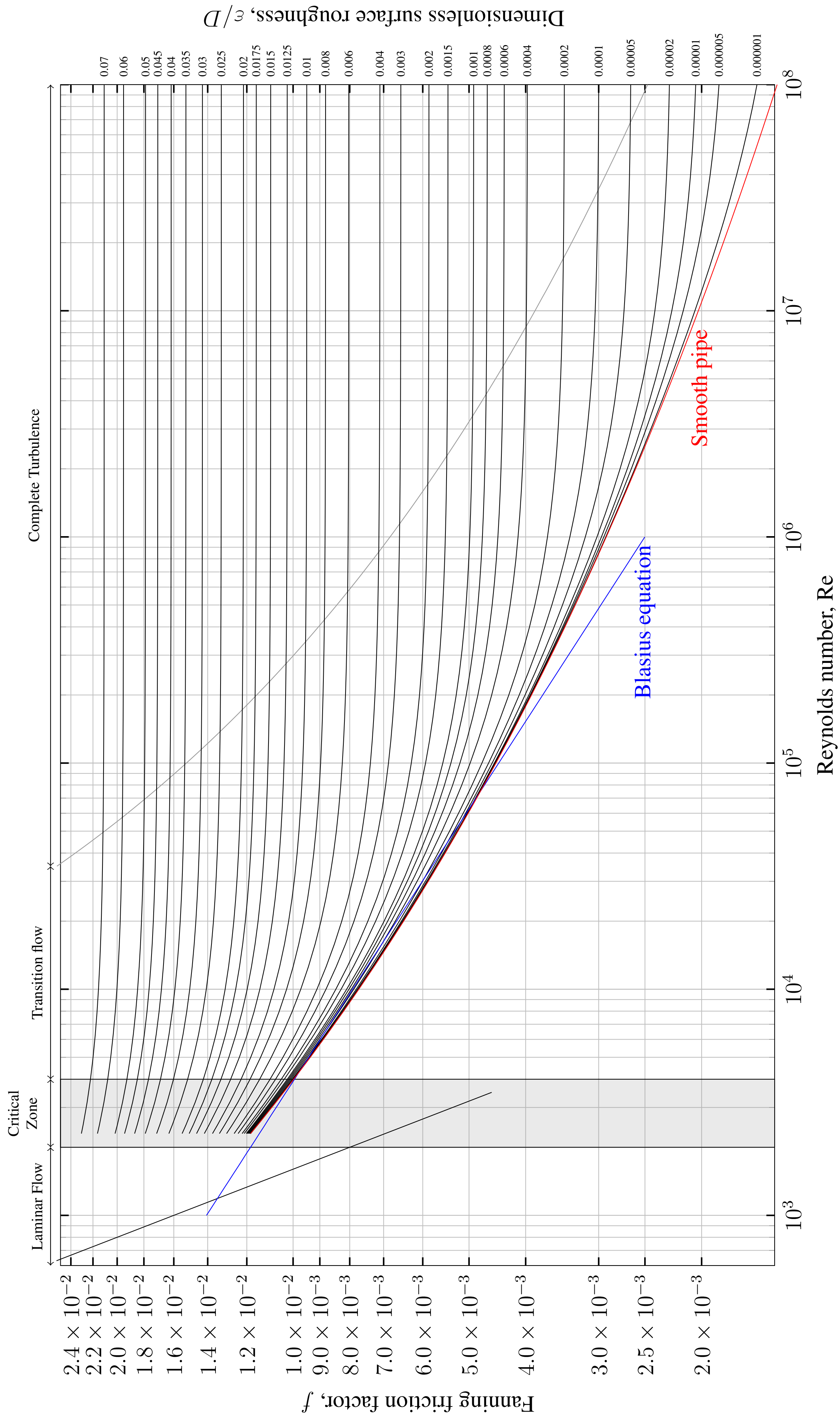

Figure 6.3: Moody chart for the fanning friction factor f.

6.5 Losses in Other Pipe Fittings

The above is useful for flow in straight lengths of pipe; however, most real systems is not just straight and has lots of appendages (e.g. valves), and it may also change diameter. The fittings have pressure losses associated with them which are caused by generally complex interplay between various factors including wall fraction, changes in direction of the flow, obstructions to the flow, and changes in the cross-sectional area and shape of the flow. For example, in a bend a secondary flow is generated where rotating motion at right angles to the pipe axis is generated which also propagates slightly downstream, so we end up with losses due to the curvature of the pipe, losses due to the bend length, and excess losses in the downstream due to the rotating flow.

Due to the complexities of the losses in fittings it is typical to group them for a given fitting into a loss coefficient in terms of the number of velocity heads in the attached pipe, K, such as [4],

$$\Delta h_f = K\frac{u^2}{2g} \tag{6.5.1}$$

Then for multiple pipe fittings in the pipe network we can add these together to get a total loss due to the fittings (often called the minor losses) as,

Minor Friction Losses

$$\Delta h_f = \sum_i K_i\frac{u_i^2}{2g} \tag{6.5.2}$$

6.5.1 Contractions and Enlargements

Taking the momentum equation it is possible to derive the form of the expressions for sudden contractions and enlargements in pipes [11]. The resistance to flow due to sudden enlargements from a small pipe of diameter D_s to a large pipe of diameter D_l may be expressed by,

$$K_E = \left(1 - \frac{D_s^2}{D_l^2}\right)^2 \tag{6.5.3}$$

and the resistance due to sudden contraction from D_l to D_s, by,

$$K_C = 0.5\left(1 - \frac{D_s^2}{D_l^2}\right) \tag{6.5.4}$$

These are losses based on the velocity in the small pipe, D_s.

The losses due to gradual enlargements in pipes were investigated by Gibson [5] and correlated to a coefficient C_E which is multiplied by the sudden enlargement number of velocity heads K_E and is dependant on the angle of divergence of the enlargement θ_E, such as,

$$C_E = \begin{cases} 2.6\sin\frac{\theta_E}{2} & \theta_E \leq 45^\circ \\ 1 & 45^\circ < \theta_E \leq 180^\circ \end{cases} \tag{6.5.5}$$

Crane [4] repeated this methodology for gradual contractions and developed the coefficient, C_C,

$$C_C = \begin{cases} 1.6 \sin \dfrac{\theta_C}{2} & \theta_C \leq 45^\circ \\ \sqrt{\sin \dfrac{\theta_C}{2}} & 45^\circ < \theta_C \leq 180^\circ \end{cases} \tag{6.5.6}$$

6.5.2 Bends and Tees

As previously mentioned bends suffer from the generation of secondary flow loops perpendicular to the flow direction. This causes increased friction losses in both the bend and the downstream pipe. The total head loss number, K_b, can be given by,

$$K_b = K_p + K_c + K_l = K_e + K_l \tag{6.5.7}$$

where K_p is the excess loss in the downstream pipe, K_c is the loss due to the curvature, and K_l is the loss due to the bend length, and K_e can be thought of as the excess loss over the bend length. Experimentally this has been examined by Beij [1] for 90° radius bends and related to the bend radius R_b to the pipe diameter D as,

$$K_b = \left(276.8 \exp\left(-1.8 \frac{R_b}{D}\right) + \frac{216.4}{1 + 6.5 \exp\left(-0.2 \frac{R_b}{D}\right)} \right) f_T \tag{6.5.8}$$

where f_T is the friction factor at complete turbulence for a pipe of the same material with the same diameter as the bend (equation 6.4.4). This is applicable for $1 \leq R_b/D \leq 20$.

For standard 90° elbows then the radius is a fixed proportion of the diameter and thus it reduces to,

$$K_b = 120 f_T$$

Standard Tees are similar to bends if the flow moves through the side branch, or like a straight pipe with side obstruction if the flow moves through the straight run,

$$K_t = \begin{cases} 80 f_T & \text{Flow through run} \\ 240 f_T & \text{Flow through branch} \end{cases}$$

6.5.3 Valves

There are a large number of different types of valves, these different types may include contractions, enlargements, changes to the flow direction, and objects within the flow. Not only this, but it can also depend on the proportion that the valve is open. Therefore, it is common to use experimental determined values for the actual valve or a similar valve, as in Table 6.2.

6.5.4 General

Due to the accuracy of the minor loss values often as a first approximation a constant is taken that is independent of the fully turbulent friction factor as shown in Appendix B.4. These tend to be based on smaller pipes where the losses are higher, so an over estimation is produced.

Table 6.2: Example K values for different types of valves.

Valve	K
Globe valve, fully open	$1360f_T$
Gate valve, fully open	$32f_T$
Gate valve, 0.25 closed	$56f_T$
Gate valve, 0.5 closed	$448f_T$
Gate valve, 0.75 closed	$3640f_T$

6.6 References

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Chapter 7

Piping Systems, Networks, and Fluid Machines

Contents

7.1 Chapter 7 ILOs

ILO 7.1. Apply Bernoulli's equation to analyse networks of pipes.

ILO 7.2. Use the integral force balance to calculate forces within pipes.

ILO 7.3. Recognise why pipes need fittings to support them.

7.2 Introduction

In the previous chapters we have looked at sections of pipes or a single pipe system. In this chapter we will now move onto more real systems with series and parallel flow and we will also thinking about how we have to support these pipes with fittings due to the forces from the fluid.

7.3 Pipe Networks

In many pipe systems we will have a network of pipes connected in many different ways, this means that we may have sections of pipe connected in series and in parallel arrangements.

7.3.1 Series Pipes

Figure 7.1 shows a very simple set of two pipes in series. If we want the flow rate in the pipes and the total pressure drop then we can simply add the losses in our two pipe section (plus the sudden contraction for a complete answer). This is what we have been doing in Chapter 6, adding the pipe losses in series.

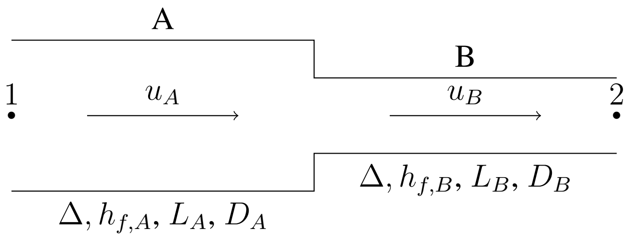


Figure 7.1: Pipes in a series arrangement.

From the mass continuity equation, equation 4.3.6 with constant density,

$$Q_A = Q_B \tag{7.3.1}$$

From Bernoulli's Equation, equation 4.4.4, the pressure drop between 1 and 2 is,

$$\Delta P_T = \Delta P_A + \Delta P_B \tag{7.3.2}$$

or with constant density,

$$\frac{\Delta P_T}{\rho g} = \frac{\Delta P_A}{\rho g} + \frac{\Delta P_B}{\rho g}$$
$$\Delta h_T = \Delta h_A + \Delta h_B \tag{7.3.3}$$

From Darcy's equation, equation 6.3.11,

$$\Delta h = \left(\sum_i K_i\right)\frac{u^2}{2g} = \frac{\sum_i K_i}{2g}\left(\frac{Q}{A}\right)^2 = \frac{\sum_i K_i}{2gA^2}Q^2 = R^*Q^2 \tag{7.3.4}$$

where K is the relevant loss parameter for fittings or $4fL/D$ for a straight pipe.

Combining equation 7.3.4 with equation 7.3.3 gives,

$$\Delta h_T = R_T^* Q^2 = R_A^* Q_A^2 + R_B^* Q_B^2 \tag{7.3.5}$$

and using equation 7.3.1 means that,

$$R_T^* Q^2 = R_A^* Q^2 + R_B^* Q^2 \tag{7.3.6}$$

so for a series system the total hydraulic resistance is,

Series Hydraulic Resistance

$$R_T^* = R_A^* + R_B^* \tag{7.3.7}$$

with,

$$\Delta P_T = \rho g R_T^* Q^2 \tag{7.3.8}$$

7.3.2 Parallel Pipes

Figure 7.2 shows a very simple set of two pipes in parallel. If we want the flow rate in the pipes and the total pressure drop then we need to understand the hydraulic resistances.

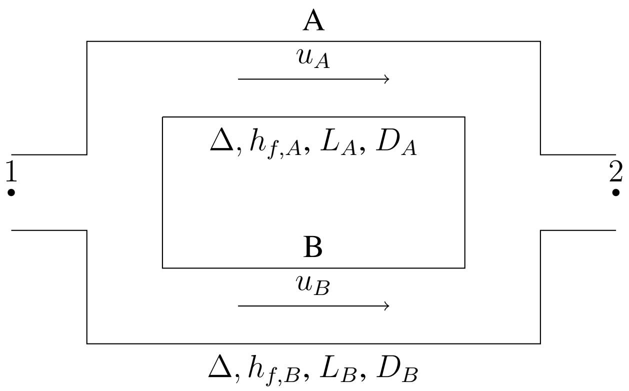


Figure 7.2: Pipes in a parallel arrangement.

From the mass continuity equation, equation 4.3.6 with constant density,

$$Q = Q_A + Q_B \tag{7.3.9}$$

From Bernoulli's Equation, equation 4.4.4, the pressure drop between 1 and 2 is,

$$\Delta P_T = \Delta P_A = \Delta P_B \tag{7.3.10}$$

or with constant density,

$$\Delta h_T = \Delta h_A = \Delta h_B \tag{7.3.11}$$

Combining equation 7.3.4 with equation 7.3.11 gives,

$$\Delta h_T = R_T^* Q^2 = R_A^* Q_A^2 = R_B^* Q_B^2 \tag{7.3.12}$$

and substituting into equation 7.3.9 means that,

$$Q = \sqrt{\frac{\Delta h_T}{R_T^*}} = \sqrt{\frac{\Delta h_T}{R_A^*}} + \sqrt{\frac{\Delta h_T}{R_B^*}} \tag{7.3.13}$$

so for a parallel system the total hydraulic resistance can be given by,

Parallel Hydraulic Resistance

$$\frac{1}{\sqrt{R_T^*}} = \frac{1}{\sqrt{R_A^*}} + \frac{1}{\sqrt{R_B^*}} \tag{7.3.14}$$

7.4 Applications of the Momentum Equation to Forces

In Chapter 6 we looked at a force balance on a fluid element as equation 6.2.1,

$$\mathbf{F}_T = -\oiint_\Gamma \rho\mathbf{V}\cdot(\mathbf{V}\cdot\mathbf{n})\,\mathrm{d}\,\Gamma - \oiint_\Gamma P\mathbf{n}\mathrm{d}\,\Gamma + \oiint_\Gamma \boldsymbol{\tau}\cdot\mathbf{n}\mathrm{d}\,\Gamma - \oiint_\Gamma \rho gz\cdot\mathbf{n}\mathrm{d}\,\Gamma \tag{7.4.1}$$

In the fluid element example the total force was equal to zero as it wasn't moving. However, in a real pipe with changes in direction or cross-sectional area there can be a total force (a fire hose needs supporting by several firefighters to stop it from flying around). This means that to stop the movement there must be a reaction force that holds the pipe in place, thus we have,

$$-\oiint_\Gamma \rho\mathbf{V}\cdot(\mathbf{V}\cdot\mathbf{n})\,\mathrm{d}\,\Gamma - \oiint_\Gamma P\mathbf{n}\mathrm{d}\,\Gamma + \oiint_\Gamma \boldsymbol{\tau}\cdot\mathbf{n}\mathrm{d}\,\Gamma - \oiint_\Gamma \rho gz\cdot\mathbf{n}\mathrm{d}\,\Gamma + \mathbf{F}_r = 0 \tag{7.4.2}$$

7.4.1 Stationary vertical column of fluid

If we take the example of a vertical column of fluid of constant cross-sectional area, Figure 7.3, then we can apply our force balance to this.

As the fluid is stationary then there is no momentum force, no shear force, and no reaction force thus our force balance, equation 7.4.2, reduces to,

$$\cancelto{0}{-\oiint_\Gamma \rho\mathbf{V}\cdot(\mathbf{V}\cdot\mathbf{n})\,\mathrm{d}\,\Gamma} - \oiint_\Gamma P\mathbf{n}\mathrm{d}\,\Gamma + \cancelto{0}{\oiint_\Gamma \boldsymbol{\tau}\cdot\mathbf{n}\mathrm{d}\,\Gamma} - \oiint_\Gamma \rho gz\cdot\mathbf{n}\mathrm{d}\,\Gamma + \cancelto{0}{\mathbf{F}_r} = 0 \tag{7.4.3}$$

This means that we have,

$$-\oiint_\Gamma P\mathbf{n}\mathrm{d}\,\Gamma = \oiint_\Gamma \rho gz\mathbf{n}\mathrm{d}\,\Gamma \tag{7.4.4}$$

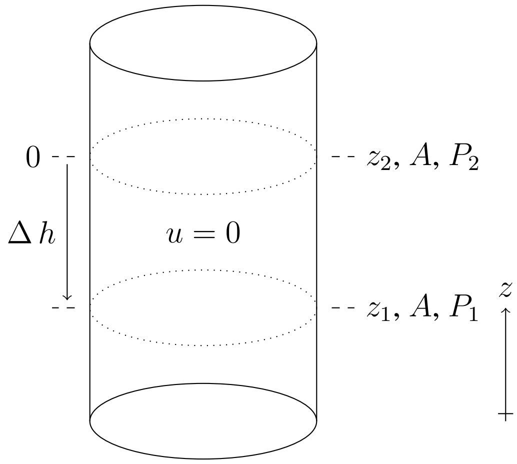


Figure 7.3: Stationary column of fluid.

Taking z to be our upwards direction, then we have the pressure on the top and bottom areas and the force due to gravity on these same areas. There is no force on the walls, thus,

$$-\left(P_1\left(-1\right)A + P_2\left(1\right)A\right) = \rho g z_1\left(-1\right)A + \rho g z_2\left(1\right)A$$
$$-\left(P_2 - P_1\right)A = \rho g\left(z_2 - z_1\right)A$$
$$P_1 - P_2 = \rho g\left(z_2 - z_1\right)$$

Therefore if we define z_2 as the base point and thus the depth of the fluid $\Delta\, h = z_2 - z_1$, then we get,

$$\Delta\, P = \rho g \Delta\, h \tag{7.4.5}$$

which is equivalent to equation 2.3.13. This means that, as it should, the Navier-Stokes' equation predicts the fluid hydrostatics.

7.4.2 Flow through a contracting section of pipework or through a nozzle

Now, let's replace our fluid element with a section of pipe of varying cross section, as shown in Figure 7.4. A control volume is defined by the dotted line, over which we can perform a force balance. Fluid flows into the control volume at point 1 with velocity u_1, and flows out at point 2 with velocity u_2.

The pipe only has flow in the x-direction, so we can look at a 1 dimensional solution to equation 7.4.2. As the pipe is horizontal, there is no gravity force in the x-direction. Also, in this case we will assume ideal frictionless flow, so there is no shear (drag) force,

$$0 = F_{m,x} + F_{p,x} + \cancelto{0}{F_{s,x}} + \cancelto{0}{F_{g,x}} + F_{r,x} \tag{7.4.6}$$

Therefore, if we know the forces due to the pressure and momentum, we can calculate the reaction force, F_{rx}, i.e. the force needed to hold the pipework in place.

The momentum force due to the fluid flowing into and out of the pipe at different velocities can be taken from equation 7.4.2. The surface integral can be split into multiple integrals,

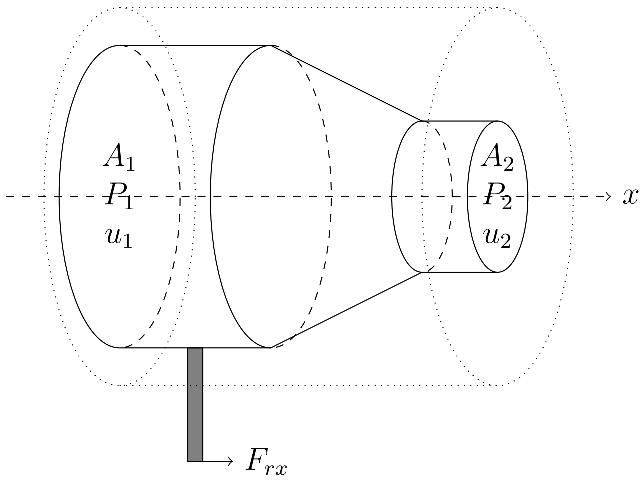


Figure 7.4: Contracting pipe, through which fluid is flowing with mass flowrate $\dot{m}$.

one for each surface of the pipe, thus,

$$\begin{aligned} F_{m,x} &= -\oiint_{\Gamma} \rho \mathbf{V} \cdot (\mathbf{V} \cdot \mathbf{n})\, \mathrm{d}\,\Gamma \\ &= -\oiint_{\Gamma_{\text{inlet}}} \rho \mathbf{V} \cdot (\mathbf{V} \cdot \mathbf{n})\, \mathrm{d}\,\Gamma_{\text{inlet}} - \oiint_{\Gamma_{\text{sides}}} \rho \mathbf{V} \cdot (\mathbf{V} \cdot \mathbf{n})\, \mathrm{d}\,\Gamma_{\text{sides}} - \oiint_{\Gamma_{\text{outlet}}} \rho \mathbf{V} \cdot (\mathbf{V} \cdot \mathbf{n})\, \mathrm{d}\,\Gamma_{\text{outlet}} \\ &= -\rho u_1 (u_{1,x} \cdot (-1)) A_1 - (0) - \rho u_2 (u_{2,x} \cdot (1)) A_2 \\ &= \rho A_1 u_1^2 - \rho A_2 u_2^2 \end{aligned} \tag{7.4.7}$$

as all the velocity is in the x-direction.

> **Constant Velocity Simplification**
>
> Due to our assumption of one-dimensional flow, we estimate $\mathbf{V} = u$. This is true for invicid flow as the flow velocity profile is flat, turbulent flow is a good approximation for this. As we have a constant velocity the surface integral can be simplified,
>
> $$\oiint_{\Gamma} \mathbf{V}\mathrm{d}\,\Gamma = \oiint_{\Gamma} u\mathrm{d}\,\Gamma = u \oiint_{\Gamma} \mathrm{d}\,\Gamma = uA$$

From mass continuity, equation 4.3.6,

$$u_2 = u_1 \frac{A_1}{A_2} \tag{7.4.8}$$

so

$$\begin{aligned} F_{m,x} &= \rho A_1 u_1^2 - \rho A_2 u_1^2 \left(\frac{A_1}{A_2}\right)^2 \\ &= \rho A_1 u_1^2 - \rho A_1 u_1^2 \frac{A_2}{A_1} \left(\frac{A_1}{A_2}\right)^2 \\ &= \rho A_1 u_1^2 \left(1 - \frac{A_1}{A_2}\right) \end{aligned} \tag{7.4.9}$$

Note that this is negative, as $A_1 > A_2$ in this case.

Consider now the overall force on the control volume due to pressure as taken from equation 7.4.2 and as P_1 and P_2 operate in the x-direction,

$$\begin{aligned}
F_{p,x} &= -\oiint_{\Gamma} P\mathbf{n}\,d\,\Gamma \\
&= -\oiint_{\Gamma_{\text{inlet}}} P\mathbf{n}\,d\,\Gamma_{\text{inlet}} - \oiint_{\Gamma_{\text{sides}}} P\mathbf{n}\,d\,\Gamma_{\text{sides}} - \oiint_{\Gamma_{\text{outlet}}} P\mathbf{n}\,d\,\Gamma_{\text{outlet}} \\
&= -P_1(-1)A_1 - P_2(1)A_2 \\
&= P_1A_1 - P_2A_2
\end{aligned} \tag{7.4.10}$$

Gauge Pressure Force

For the pressure force analysed in this manner it is important to use the Gauge Pressure. This is because atmospheric pressure is acting on every surface equally so it cancels out. The gauge pressure is therefore the "excess" pressure above this atmospheric pressure. This of course assumes our pipe system is sitting in atmospheric pressure as most of the systems we will deal with are.

Let's assume, for the sake of this example, that the jet is exiting at point 2 to atmospheric pressure, such that P_2 is zero (This would describe a nozzle on the end of a hose, for example). Then the pressure force is simplifed to,

$$F_{p,x} = P_1A_1 \tag{7.4.11}$$

From Bernoulli's Equation, equation 4.4.4,

$$\begin{aligned}
P_1 + \frac{\rho u_1^2}{2} &= \cancelto{0}{P_2} + \frac{\rho u_2^2}{2} \qquad\qquad u_2^2 = u_1^2\left(\frac{A_1}{A_2}\right)^2 \\
P_1 + \frac{\rho u_1^2}{2} &= \frac{\rho}{2}u_1^2\left(\frac{A_1}{A_2}\right)^2 \\
P_1 &= \frac{\rho}{2}u_1^2\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right) \\
P_1A_1 &= \frac{\rho}{2}A_1u_1^2\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right) \\
F_{p,x} &= \frac{\rho}{2}A_1u_1^2\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right) \\
&= \frac{\rho}{2}A_1u_1^2\left(\frac{A_1}{A_2} - 1\right)\left(\frac{A_1}{A_2} + 1\right)
\end{aligned} \tag{7.4.12}$$

Note that this is positive as $A_1 > A_2$ in this case. The pressure force pushes on the nozzle in the positive x-direction, trying to push if off the end of the hose.

So, the reaction force in the x-direction is given by,

$$\begin{aligned}
F_{r,x} &= -F_{m,x} - F_{p,x} \\
&= -\rho A_1u_1^2\left(1 - \frac{A_1}{A_2}\right) - \frac{\rho}{2}A_1u_1^2\left(\frac{A_1}{A_2} - 1\right)\left(\frac{A_1}{A_2} + 1\right) \\
&= \rho A_1u_1^2\left(\frac{A_1}{A_2} - 1\right) - \frac{\rho}{2}A_1u_1^2\left(\frac{A_1}{A_2} - 1\right)\left(\frac{A_1}{A_2} + 1\right)
\end{aligned} \tag{7.4.13}$$

The reaction force is negative overall. This is because the pressure force is larger than the momentum force. By expressing both forces in terms of A_1 and u_1, it is possible to compare the two forces directly and hence to see this. From equation 7.4.13, in which the first term is the momentum force and the second is the pressure force, clearly the latter is larger than the former because $(A_1/A_2 + 1) > 2$, if $A_1 > A_2$.

By collecting the like terms together, we can simplify the equation further:

$$\begin{aligned} F_{r,x} &= \rho A_1 u_1^2 \left(\frac{A_1}{A_2} - 1 \right) \left(2 - \left(\frac{A_1}{A_2} + 1 \right) \right) \\ &= \rho A_1 u_1^2 \left(\frac{A_1}{A_2} - 1 \right) \left(1 - \frac{A_1}{A_2} \right) \\ &= -\rho A_1 u_1^2 \left(\frac{A_1}{A_2} - 1 \right)^2 \\ &= -\rho A_1 \left(u_2 - u_1 \right)^2 \end{aligned} \quad (7.4.14)$$

Again, it is evident from this final equation that the reaction force is negative with respect to the x-direction as $u_2 > u_1$.

Note that the analysis applies equally if $A_2 > A_1$. And the reaction force would still be negative! This seems strange! It occurs because the pressure and momentum forces both reverse sign but also swap relative magnitudes. Equivalently, if the direction of flow is reversed, the reaction force is still in the negative x-direction, for the same reason.

7.4.3 Flow around a contracting pipe bend

Figure 7.5 shows a pipe bend with a reducing diameter from A_1 at the inlet to A_2 at the outlet. The gauge pressure at the inlet is P_1 and the volumetric flowrate of water is Q As the fluid flows around the bend, it will exert a force which will need to be counterbalanced in order to hold the pipework in place. Ignoring friction forces, and assuming the pipe bends in the horizontal plane, develop an expression for the magnitude and direction of the force exerted by the fluid on the pipe.

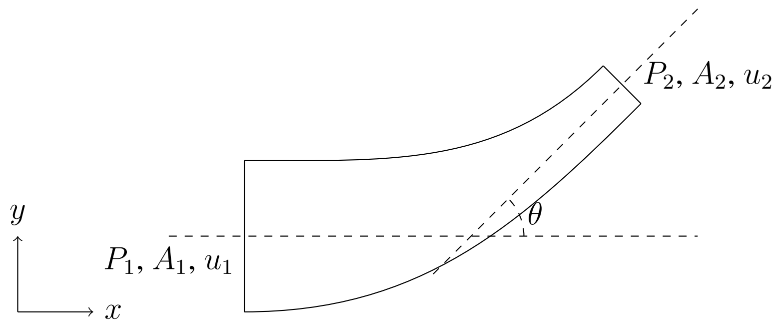


Figure 7.5: Flow through a reducing pipe bend.

As before, excluding gravity and shear forces (equation 7.4.6), and performing the force balance, but in 2 dimensions,

$$\begin{pmatrix} F_{p,x} \\ F_{p,y} \end{pmatrix} + \begin{pmatrix} F_{m,x} \\ F_{m,y} \end{pmatrix} + \begin{pmatrix} F_{r,x} \\ F_{r,y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (7.4.15)$$

At point 1, the velocity is equal to,

$$u_1 = \frac{Q}{A_1}$$

At point 2, from the conservation of mass, we can write the velocity in terms of u_1,

$$u_2 = \frac{Q}{A_2} = u_1 \frac{A_1}{A_2}$$

At point 1 the velocity is in the x-direction while at point 2 it is equal to $u_2 \times \cos\theta$ (i.e. the x component of the velocity vector). Therefore, the momentum force on the pipe bend in the x-direction, following the method of equation 7.4.7, is,

$$\begin{aligned} F_{mx} &= -\rho u_1 \left(u_{1,x} \cdot (-1)\right) A_1 - \rho u_2 \left(u_{2,x} \cdot (1)\right) A_2 \\ &= \rho A_1 u_1 u_1 - \rho A_2 u_2 \left(u_2 \cos\theta\right) \\ &= \rho A_1 u_1^2 - \rho A_2 u_2^2 \cos\theta \\ &= \rho A_1 u_1^2 - \rho A_2 \left(u_1 \frac{A_1}{A_2}\right)^2 \cos\theta \qquad (7.4.16) \\ &= \rho A_1 u_1^2 \left(1 - \frac{A_1}{A_2}\cos\theta\right) \end{aligned}$$

Similarly the momentum force in the y-direction is,

$$\begin{aligned} F_{my} &= -\rho u_1 \left(u_{1,y} \cdot (-1)\right) A_1 - \rho u_2 \left(u_{2,y} \cdot (1)\right) A_2 \\ &= \rho A_1 u_1 (0) - \rho A_2 u_2 \left(u_2 \sin\theta\right) \\ &= 0 - \rho A_2 u_2^2 \sin\theta \\ &= -\rho A_1 u_1^2 \frac{A_1}{A_2} \sin\theta \qquad (7.4.17) \end{aligned}$$

Turning now to the pressure force, from Bernoulli's equation,

$$\begin{aligned} P_2 &= P_1 + \frac{\rho}{2}\left(u_1^2 - u_2^2\right) \\ &= P_1 + \frac{\rho}{2} u_1^2 \left(1 - \frac{A_1^2}{A_2^2}\right) \qquad (7.4.18) \end{aligned}$$

This means that the pressure force in the x-dimension, following the method of equation 7.4.10, is,

$$\begin{aligned} F_{px} &= -P_{1,x}(-1)A_1 - P_{2,x}(1)A_2 \\ &= P_1 A_1 - A_2 \left(P_2 \cos\theta\right) \\ &= P_1 A_1 - P_1 A_2 \cos\theta - \frac{\rho}{2} u_1^2 A_2 \cos\theta \left(1 - \frac{A_1^2}{A_2^2}\right) \\ &= P_1 \left(A_1 - A_2 \cos\theta\right) - \frac{\rho}{2} u_1^2 A_1 \cos\theta \left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) \qquad (7.4.19) \end{aligned}$$

and in the y-dimension is,

$$\begin{aligned} F_{py} =& -P_{1,y}(-1)A_1 - P_{2,y}(1)A_2 \\ =& \,(0)A_1 - A_2\left(P_2 \sin\theta\right) \\ =& \,P_1 A_2 - A_2 \frac{\rho}{2} u_1^2 \left(1 - \frac{A_1^2}{A_2^2}\right) \\ =& -P_1 A_2 \sin\theta - \frac{\rho}{2} u_1^2 A_1 \sin\theta \left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) \end{aligned} \tag{7.4.20}$$

Don't forget that we need to use the gauge pressure, as this is the excess pressure over the atmospheric pressure which is acting on every surface equally.

Therefore from equation 7.4.15 the total force balance is,

$$\begin{pmatrix} P_1\left(A_1 - A_2\cos\theta\right) - \frac{\rho}{2}u_1^2 A_1 \cos\theta \left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) \\ -P_1 A_2 \sin\theta - \frac{\rho}{2}u_1^2 A_1 \sin\theta\left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) \end{pmatrix} + \begin{pmatrix} \rho A_1 u_1^2 \left(1 - \frac{A_1}{A_2}\cos\theta\right) \\ -\rho A_1 u_1^2 \frac{A_1}{A_2}\sin\theta \end{pmatrix} + \begin{pmatrix} F_{r,x} \\ F_{r,y} \end{pmatrix} = 0 \tag{7.4.21}$$

Therefore,

$$F_{r,x} = \frac{\rho}{2}u_1^2 A_1 \cos\theta\left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) - P_1\left(A_1 - A_2\cos\theta\right) - \rho A_1 u_1^2\left(1 - \frac{A_1}{A_2}\cos\theta\right) \tag{7.4.22}$$

and,

$$F_{r,y} = P_1 A_2 \sin\theta + \frac{\rho}{2}u_1^2 A_1 \sin\theta\left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) + \rho A_1 u_1^2 \frac{A_1}{A_2}\sin\theta \tag{7.4.23}$$

The total net force is found by the magnitude of the two force vectors, as,

$$|\mathbf{F}_r| = \sqrt{F_{r,x}^2 + F_{r,y}^2} \tag{7.4.24}$$

and the direction from,

$$\theta = \tan^{-1}\frac{F_{r,y}}{F_{r,x}} \tag{7.4.25}$$

7.5 Pipe Supports

In the last section (Section 7.4) we looked at forces on pipes as the fluid flows. It is therefore important that the supports for pipes are strong and frequent enough to stop the pipe from moving. As well as this, it is important that the weight of piping running over long or medium distances in a process plant (mass of pipe and internal fluid) and the related piping components is supported (e.g. valves, fittings, sensors). Ideally, with the supports, we want to,

- Minimise stresses in the piping
- Maintain the intended layout and slope
- Avoid excessive sag
- Minimise reaction loads on equipment nozzles
- Optimise the type, size, and location of pipe supports

This is complicated due to the fact that the pipework system must be sufficiently flexible to accommodate the movements of the components as they expand. All pipes will be installed at ambient temperature, but pipes may carry hot fluids such as water or steam. Therefore, they expand, especially in length, with an increase from ambient to working temperatures. Without some flexibility int he system this will create stress upon certain areas, such as pipe joints, which, in the extreme, could fracture.

In many cases the natural flexibility of the pipework system, due to the length of the pipe and number of bends and supports, means that no undue stresses are imposed. However, in some situations, for example long straight pipes, it will be necessary to incorporate some means of achieving this required flexibility, e.g. expansion loops.

7.6 Transient Forces

As well as looking at the static forces exerted on pipe supports, there can be situations where the force may change with time, for example the tank with variable fluid height in Section 5.3.2, at the start time there would be a large force on the tank as it is full of water, and there would also be a variable force on the exit as the fluid velocity changes. Most situations like this have very manageable changes in the force, however; there are some situations where a transient force can be very large.

One example of this is pressure surging in a pipeline. Surge pressures, sometimes called water hammers, occur in a pipe transporting fluid as a result of a change in the flow velocity, e.g. if a valve is closed or opened too rapidly. If a valve is closed rapidly, the energy of the fluid moving forward in the pipe leads to a sudden rise in pressure due to compression upstream of the valve. Negative pressure will develop immediately downstream of the valve as a result of the fluid being still in motion, causing a temporary separation of the liquid column followed by a reverse flow back towards the valve if the pressure gradient is high enough. This may lead to the destruction of both the valve and the pipe. The highest surge pressures often occur when the maximum number of pumps in operation at a pumping station stop as a result of a power failure. As the change in pressure and velocity is not limited to the point of disturbance, it continues up- and down-stream at pressure wave propagation velocity.

The momentum of a liquid in a length L of pipe diameter D is equal to the mass of fluid times the average fluid velocity u,

$$\rho\frac{\pi}{4}LD^2u \tag{7.6.1}$$

If a valve is gradually closed the momentum of the liquid will be “destroyed” and this requires that the valve exert a force on the liquid. This will therefore appear as a rise in pressure, equation 6.2.1. If the valve is closed over a period of time t then the pressure rise will be,

$$\frac{\pi}{4}\frac{\rho L D^2 u}{t} = \Delta P \frac{\pi}{4} D^2$$
$$\Delta P = \frac{\rho L u}{t} \tag{7.6.2}$$

In this case, if water is flowing with velocity $2\,\mathrm{m\,s^{-1}}$ in a pipe of diameter 0.3 m and length 1000 m and then a valve is closed over a period of 10 s then the pressure rise would be approximately 2 bar. If the valve is closed more quickly the pressure rise will be much larger. It might be thought that is a valve were to be closed instantly the pressure rise would be infinite. This is not the case the rise in pressure is actually limited by the speed a pressure wave can propagate through the fluid, which is the speed of sound in that fluid. There is a slight further reduction due to expansion of the pipe due to its elasticity, which means the actual pressure rise is given by [2],

$$\Delta P = u K^{1/2} \rho^{1/2} \left(1 + \frac{K}{E}\frac{D}{t_w}\right)^{-1/2} \tag{7.6.3}$$

where K is the bulk modulus of elasticity of the fluid in Pa, E is the bulk modulus of elasticity of the pipe material, and t_w is the pipe wall thickness.

Taking the above example with quick valve closing in a steel pipe with thickness 10 mm and $E = 2 \times 10^5$ MPa (Water, $K = 2 \times 10^3$ MPa), then the potential pressure rise could be almost 25 bar which could cause major damage to the valve and pipe.

Engineers have several options when designing piping systems to help minimise the negative impact of surge pressure. Properly sizing pipes, for example, is the most effective way to control fluid velocity. The larger the pipe diameter, the slower the fluid velocity for a given volumetric flow rate. This variable should be adjusted to maintain the required flow rate while keeping surge pressure below 1.5 times the piping material's maximum working pressure.

Linear liquid velocity within a piping system should generally be limited to between $0.6\,\mathrm{m\,s^{-1}}$ and $1.2\,\mathrm{m\,s^{-1}}$ [1] especially for pipes six inches or larger (there are exceptions to this, e.g. in small heat exchanger pipes typically up to $3\,\mathrm{m\,s^{-1}}$ is used, and in slurry systems were higher velocity is needed to suspend the solids). Maximum gas velocity is typically calculated based on allowable pressure drop due to the lower density, but it would not be uncommon to have velocities of $20\,\mathrm{m\,s^{-1}}$.

Extra protective equipment may be used to prevent surge pressure or water hammer in pipes. Such equipment might include pressure relief valves, shock absorbers, surge arrestors, and air vacuum relief valves. Fast-acting valves should always be regulated to help prevent hydraulic shock.

7.7 References

[1] Austin, L. M., ed. [2005], *Guidelines for Human Settlement Planning and Design*, CSIR Building and Construction Technology.

[2] Holland, F. and Bragg, R. [1995], *Fluid Flow for Chemical Engineers*, Butterworth-Heinemann.

Chapter 8

Pumps and Pumping

Contents

8.1 Chapter 8 ILOs

ILO 8.1. Recall different types of pumps and their uses.

ILO 8.2. Calculate piping system curves and pump opperating points.

ILO 8.3. Compare and assess practical considerations for pumps.

8.2 Introduction

We have seen in previous chapters that energy in pipes is lost to friction and that a pressure and/or height differential is needed to make fluids flow. This is not always practical as sometime we need to pump fluid 'up-hill', in this case we need to generate pressure to allow the fluid to flow. Pumps can be used to increasing the pressure of the fluid to allow it to flow along the piping system. Before you purchase a pump, you must specify the type of pump and make sure it is capable of delivering the required flowrate at the required pressure.

Pumps come in a variety of sizes for a wide range of applications. They can be classified according to their basic operating principle as (roto-)dynamic or displacement pumps.

8.2.1 (Roto-)dynamic Pumps

In a rotodynamic pump, a rotating impeller imparts energy to the fluid. The most common type of rotodynamic pump is the centrifugal pump as can be seen in Figure 8.1(a). In a centrigugal pump the liquid enters the center of the pump and is then accelerated outwards towards the edge of the impeller blades and towards the pump exit. The amount of liquid that passes through the pump is inversely related to the pressure at the pump outlet. In other words, the outlet flowrate of a rotodynamic pump varies nonlinearly with pressure.

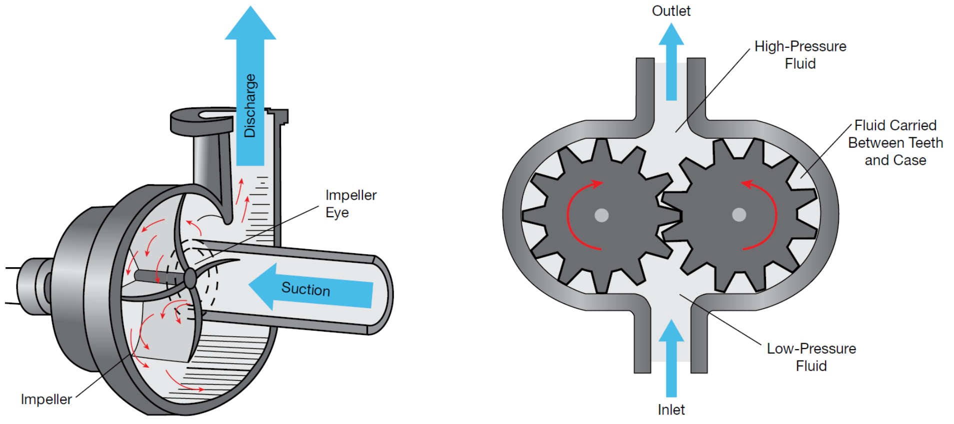
Where different pump designs could be used, the centrifugal pump is generally the most economical followed by rotary and reciprocating pumps (both displacement pumps). worldwide, centrifugal pumps account for the majority of electricity used by pumps in chemical plants.

8.2.2 Displacement Pumps

Displacement pumps can be sub-classified as rotary or reciprocating pumps. In a positive-displacement pump, a discrete amount of fluid is trapped, forced through the pump, and discharged. A gear pump is an example of a positive-displacement pump as seen in Figure 8.1(b). In a gear pump the liquid enters one side, then the rotating teeth trap some of the fluid between it and the wall, this fuild is then pushed around the pump and forced into the exit region.

This pumping principle produces a pulsating flow, rather than a smooth flow. However, the output flow tends to vary little with respect to the pressure at the pump outlet, because the moving displacement mechanism pushes the slug of liquid out at a constant rate.

In principle, any liquid can be handled by any of the pump designs. Although, positive displacement pumps are generally more efficient than centrifugal pumps especially for low flow rates at high pumping heads, the benefit of the higher efficiency tends to be offset by increased maintenance costs.



(a) (b)

Figure 8.1: Types of typical pumps used, (a) centrifugal pump, and (b) gear pump.

8.3 System Characteristics

As mentioned above, pressure is needed to make the liquid flow at the required rate and this must overcome head ’losses’ in the system. These losses can be divided into two types: static and dynamic heads. The static head is simply the difference in height and pressure of the destination and supply reservoirs. Static head is independent of flow rate. A system pumping at very low velocity or a short pipe into a closed pressurised vessel would only have a static head.

The dynamic head (sometimes called friction head loss) is the friction loss on the liquid being moved through the pipes, valves, and equipment in the system. As seen in Chapter 6, the friction losses are proportional to the square of the flow rate. A closed loop circulating system without a surface open to atmospheric pressure, would exhibit only a dynamic head.

Most systems have a combination of static and dynamic head and the system curves can be developed from Bernoulli’s equation 4.5.7 with the addition of a term for what head would need to be added from a pump, Δh_p, for the required flow.

$$\frac{P_1}{\rho_1 g} + \frac{u_1^2}{2g} + h_1 + \Delta h_p = \frac{P_2}{\rho_2 g} + \frac{u_2^2}{2g} + h_2 + \Delta h_f \tag{8.3.1}$$

This can then be rearranged for the required pump head as the system curve,

System Curve

$$\Delta h_p = \underbrace{\frac{P_2 - P_1}{\rho g} + (h_2 - h_1)}_{\text{Static Head}} + \underbrace{\frac{u_2^2 - u_1^2}{2g} + \Delta h_f}_{\text{Dynamic Head}} \tag{8.3.2}$$

For a fluid being pumped from a large vessel (i.e. $u_1 \approx 0$) in a single diameter circular pipe to an exit, the system curve can be simplified to,

$$\Delta h_p = \frac{P_2 - P_1}{\rho g} + (h_2 - h_1) + \frac{u_2^2}{2g} + \sum_i K_i \frac{u_2^2}{2g}$$

$$\Delta h_p = \frac{P_2 - P_1}{\rho g} + (h_2 - h_1) + \left(1 + \sum_i K_i\right) \frac{u_2^2}{2g}$$

$$\Delta h_p = \frac{P_2 - P_1}{\rho g} + (h_2 - h_1) + \left(1 + \sum_i K_i\right) \frac{8Q^2}{\pi^2 g D^4}$$

where K are the friction loss coefficients for the fitting and pipe (with the pipe given by $K = 4fL/D$). An example of this curve can be seen in Figure 8.2.

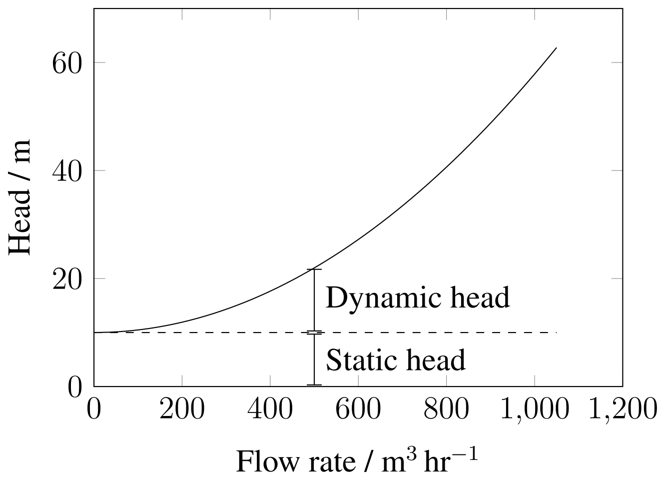


Figure 8.2: Example system curve.

Static head is a characteristic of the specific installation and reducing this head where possible, generally helps both the cost of the installation and the cost of pumping the liquid. Dynamic head losses must be minimised to reduce pumping cost, but after eliminating unnecessary pipe fittings and length, further reduction in friction head will require larger diameter pipes, which adds to installation cost.

8.4 Pump Curves

The performance of a pump can be expressed graphically as the outlet head against the flow rate produced. The most frequent use of pump curves is in the selection of centrifugal pumps, as the flowrate of these pumps varies dramatically with system pressure. Pump curves are used far less frequently for positive-displacement pumps.

On a typical pump curve, the flowrate (Q) is on the horizontal axis and head (Δh_p) is on the vertical axis. The intersection of the curve with the vertical axis corresponds to the closed valve head of the pump (i.e. zero flowrate). These curves are generated by the pump manufacturer under shop test conditions and ideally represent the average values for a representative sample of pumps.

It is common to have the pump efficiency, the power, and the net positive suction head (NPSH) plotted on the same graph as the pump curve, Figure 8.3.

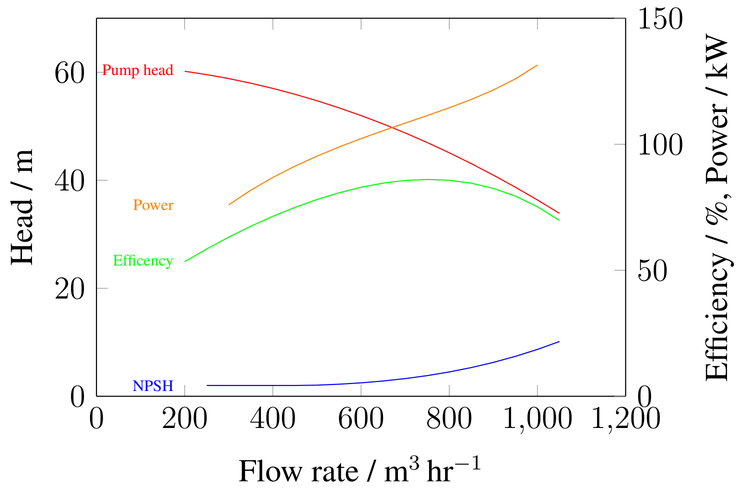


Figure 8.3: Pump pertormance curve.

The pump efficiency, eta, is the ratio between the shaft power $\dot{W}$, i.e. that imparted to the fluid, to the total power needed to run the pump, $\dot{W}_R$. If the efficiency is not know then a good approximation is $\eta = 0.7$.

This total power is given by,

Pump Power

$$\dot{W}_R = \frac{Q\rho g\Delta h_p}{\eta} \tag{8.4.1}$$

The pump manufacturer will provide the precise power ratings and motor size for the pump when ordering, but the electrical engineers need an approximate value of this (and pump location) early in the design process to allow them to size the power cables. You should err on the side of caution in this rating calculation as the electrical engineers will be much happier if you come back later to ask for a lower power rating than a higher one.

8.4.1 Suction Head and Net Positive Suction Head

The Net Positive Suction Head is a measure of the pressure experienced by a fluid on the suction side of a centrifugal pump. It is defined as the total head of fluid at the centre line of the impeller less the fluid's vapour pressure. Thus the purpose of NPSH is to identify and avoid the operating conditions which lead to vaporisation of the fluid as it enters the pump – a condition known as flashing. In a centrifugal pump, the fluid's pressure is at a minimum at the centre of the impeller. If the pressure here is below the vapour pressure of the fluid, bubbles are formed which pass on through the impeller blades towards the discharge port. As the bubbles of vapour are transported into this higher pressure region, they can spontaneously collapse in a damaging process called cavitation. The repeated shock waves produced by this process can be a significant cause of wear and metal fatigue on impellers and pump cases. Cavitation also results in vibration and noise in the pump, placing greater strain on the drive shaft and other components, and also in the downstream pipework. This can lead to greater maintenance costs and a higher incidence of pump failures.

NPSH is normally considered in two forms: NPSH-R (NPSH Required) and NPSH-A (NPSH Available).

The NPSH-R is a pump property and is the value shown on pump curves, Figure 8.3. NPSH-R is quoted by pump manufacturers as a result of extensive testing under controlled conditions. NPSH-R is a minimum suction pressure that must be exceeded for the pump to operate correctly and minimise flashing and cavitation. Manufacturers test pumps under conditions of constant flow and observe the discharge pressure as NPSH (the suction pressure) is gradually reduced. NPSH-R is defined as the value at which the discharge pressure is reduced by 3% because of the onset of cavitation. Tests are usually performed with water at $20\,°\mathrm{C}$.

The NPSH-A is a system property and needs to be calculated from the suction-side system configuration. It is essentially the suction-side pressure less the vapour pressure of the pumped fluid at that point. The NPSH-A must exceed the pump's NPSH-R rating for the chosen operating conditions to ensure that cavitation is avoided. Normally, a safety margin of 0.5 to 1 m is required to take account of this and other factors such as the pump's operating environment, changes in the weather, and any increases in friction losses that may occur occasionally or gradually during the lifetime of the system.

The NPSH-A is calculated from the suction side configuration taking into account friction losses and the vapour pressure of the pumped fluid as,

Net Positive Suction Head Avaliable

$$\text{NPSH-A} = \frac{P_o}{\rho g} + h_o - \Delta\, h_{f,s} - \frac{P^o}{\rho g} \tag{8.4.2}$$

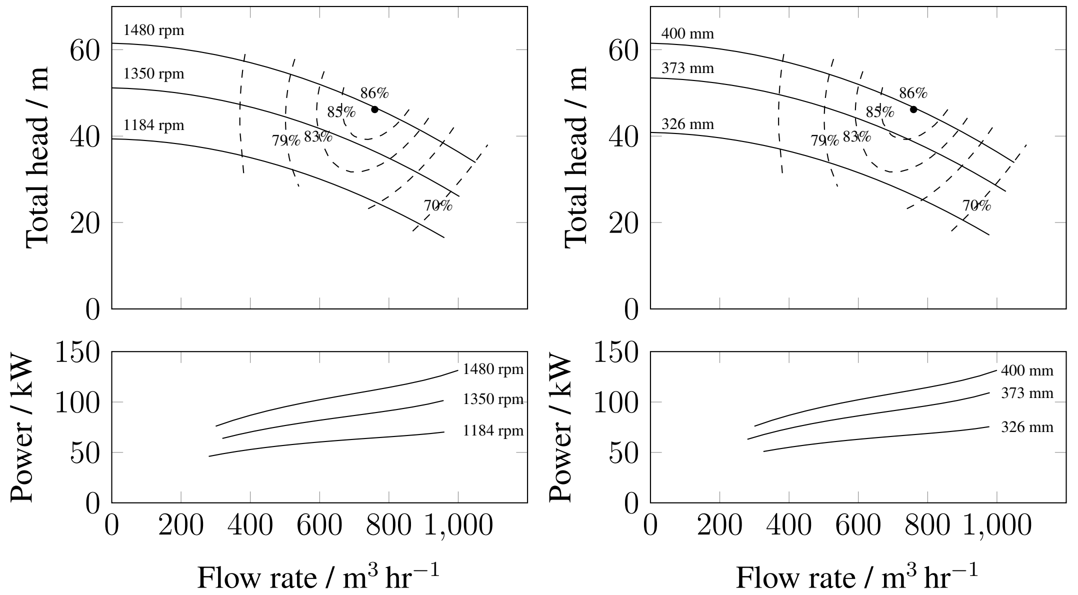
where P_o is the absolute pressure at the suction side reservoir, h_o is the reservoir liquid level relative to the pump centerline, $\Delta\, h_{f,s}$ is the headloss due to friction on the suction side of the pump, and P^o is the vapour pressure of the pumped fluid. Note that NPSH is calculated differently for centrifugal and positive-displacement pumps, and that it varies with pump speed for positive-displacement pumps rather than with pressure as for centrifugal pumps as shown here.

8.4.2 Effect of pump parameters

More sophisticated curves may include nested curves representing the flow/head relationship at different supply frequencies (i.e., the AC electrical supply's frequency in Hz), at rotational speeds (e.g. Figure 8.4(a)), with different impeller sizes (e.g. Figure 8.4(b)), or for different fluid densities. As can be seen in Figure 8.4, curves for larger impellers or faster rotation speeds lie above curves for smaller impellers or slower rotation as they are imparting more energy to the fluid. Curves for lower-density fluids lie above curves for higher-density fluids as the same volumetric flow rate for higher density fluids contains more mass.

Corresponding power curves for each impeller are shown on the bottom of the figure and the dashed lines in Figure 8.4 are efficiency curves. These curves can start to look a bit confusing, but the important point to keep in mind is that, just as in the simpler examples, flowrate is always on a common horizontal axis, and the corresponding value on any curve is vertically above or below the duty point.

These more-advanced curves usually incorporate efficiency curves, and these curves define a region of highest efficiency. At the center of this region is the best efficiency point (BEP). The NPSH may also be shown on these multiple curves.



(a) 400 mm (b) 1480 rpm

Figure 8.4: Typical centrifugal pump performance curves for (a) variation of speed at a fixed blade size and (b) variation of blade size at a fixed speed.

8.5 Pump Operating Points

The system head curve (equation 8.3.2) can be plotted on the same axes as the pump curve. The point at which the system curve and the pump curve intersect is the operating point, or duty point, of the pump, Figure 8.5. The optimal point for the pump duty point is the best efficiency point (BEP), i.e. the peak of the efficiency curve. If the duty point is far to the right of a pump curve, well away from the BEP, it is not the right pump for this system.

Remember that a system curve applies to a range of flows at a given system configuration. Throttling a valve in the system will produce a different system curve. If flow through the system will be controlled by opening and closing valves, you need to generate a set of curves that represent expected operating conditions, with a corresponding set of duty points.

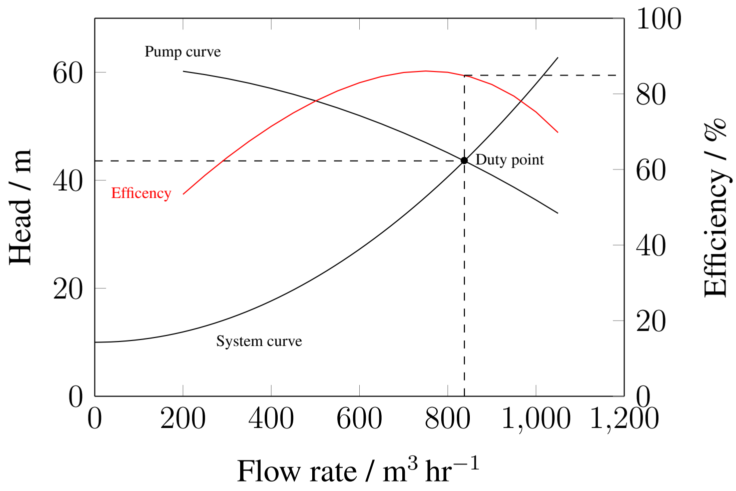


Figure 8.5: Plot to calculate the pump duty point.

8.6 Towards Real Systems

The previous sections describe how to calculate the headloss through a single line, but what about the common situation where the process has branched lines, manifolds, multipl pumps, and so on? When each branch handles a flow proportional to its headloss, and its headloss is proportional to the flow passing through it, producing an accurate model can become complex very quickly.

Typically software can be used to perform full hydrodynamic calculations for the full plant, for example solving the Hardy Cross method which is a method for determining the flow in a pipe network when the flows within the network are unknown but the inputs and outputs are known [1]. However, initial design phases will require simplified calculations for pump specifications. One method is to perform headloss calculations for each section of the simplified plant design at expected flows to find the flow path with the highest headloss. Use the highest-headloss path to determine the required pump duty and calculate the pump duty at both the average flow with working flow equalisation, and at full flow through the single branch. Usually these do not differ much, and the true answer lies between them.

Sometimes multiple pumps are needed either because there are large changes in the flow rate needed, the flow rate is very high, or the head required is very high. Pumps are used in series to increase the head, Figure 8.6(a), the head produced by two pumps is the combined head of both pumps, with the same flow rate. Pumps are used in parallel to increase the flow rate, Figure 8.6(b), the flowrate produced by two pumps is the combined flowrate of both pumps, with the same pump head.

If multiple pumps are used for flow control as two or more pumps to operate in parallel (often used for systems where static head is a high proportion of the total head) then variation of the flowrate is achieved by switching on and off additional pumps to meet demand. Care must be taken when running pumps in parallel to ensure that the operating point of the pump is controlled within the region deemed as acceptable by the manufacturer. For example, if the second pump in Figure 8.6(b) is stopped then the remaining pump could operate well out along the curve where the efficiency is lower and where the NPSH is higher and vibration level increased, giving an increased risk of operating problems.

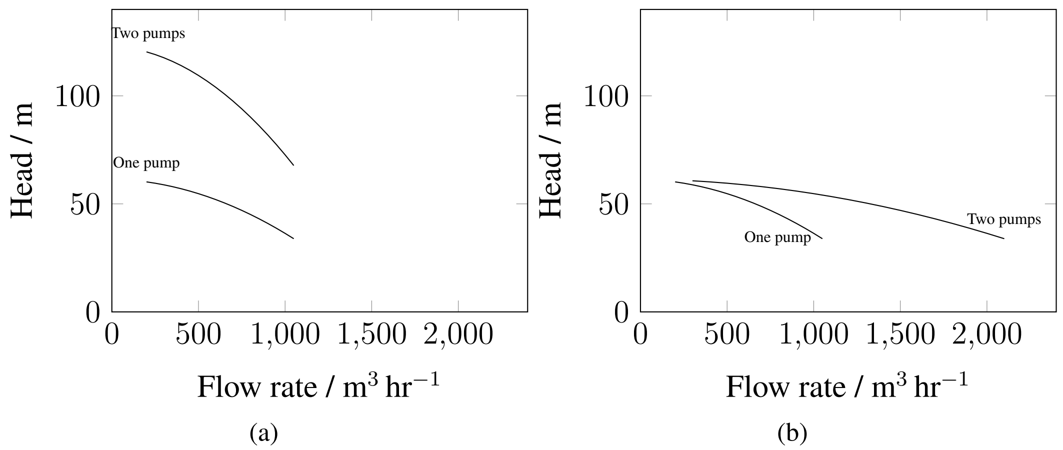


Figure 8.6: Effect of multiple pumps (a) series and (b) parallel.

8.7 References

[1] Huddleston, D. H., Alarcon, V. J. and Chen, W. [2004], 'A spreadsheet replacement for hardy-cross piping system analysis in undergraduate hydraulics', *Critical Transitions in Water and Environmental Resources Management* pp. 1–8.

Chapter 9

Open-Channel Flow

Contents

9.1 Chapter 9 ILOs

ILO 9.1. Understand how flow in open channels differs from flow in pipes.

ILO 9.2. Calculate the velocity profile for laminar flow.

ILO 9.3. Determine the flow height in open-channels of different shapes.

9.2 Introduction

Open-channel flows are those that are not entirely included within rigid boundaries; a part of the flow is in contract with nothing at all, just empty space (though really a gas), Figure 9.1(a). The surface of the flow thus formed is called a free surface, because that flow boundary is freely deformable, in contrast to the solid boundaries. The boundary conditions at the free surface of an open-channel flow are always that both the pressure and the shear stress are zero everywhere.

A flow can have a free surface but not be an open-channel flow. Closed-conduit flows that consist of two immiscible fluid phases of differing density in contact with each other along some bounding surface are not open-channel flows, because they are nowhere in contact with open space, but they do have a freely deformable boundary within them. Such flows are free-surface flows but not open-channel flows, Figure 9.1(b), although they are usually called stratified flows, because the density difference between the two fluids gives rise to gravitational effects in the flow.

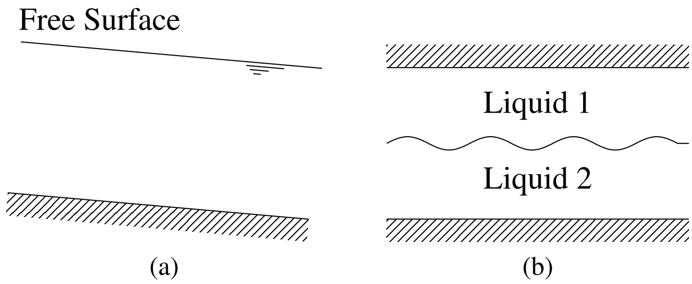


Figure 9.1: (a) Open-channel and (b) free-surface flow that is not an open-channel.

All of the principles and techniques for dealing with velocity structure and boundary resistance that were developed for pipe flows in the earlier chapters hold as well for open-channel flows. But open-channel flows involve an important added element of complexity beyond what we have covered on laminar and turbulent flows in pipes: the presence of the free surface means that the geometry of the flow can change in the flow direction not just by being constrained to do so by virtue of the geometry of the boundaries but also by the behaviour of the flow itself. This means that the acceleration of gravity can no longer be ignored by the expedient of subtracting out the hydrostatic pressure, as with pipe flows, because the force of gravity helps to shape the free surface. So gravity must therefore be included as an additional independent variable in dealing with free-surface flows.

There are generally two types of flow in open-channel systems, uniform flow and non-uniform flow. A non-uniform flow is one in which the fluid velocity and fluid depth vary over distance, while in uniform flow they remain constant. Uniform flow can occur only in a channel of constant cross-section, roughness, and slope in the flow direction; however, non-uniform flow can occur in such a channel or in a channel with variable properties.

Also, under the right conditions gravity waves can be generated on the free surface, whether or not the fluid is flowing. When the deformable free surface is momentarily deformed in some small area by a deforming force, for example, the force of the wind, or by agitating the water with something entering the water as the force of gravity acts to try to restore the free surface to its original planar condition. Provided that the viscosity of the liquid is not too high this attempt at restoration of a deformed free surface leads to the propagation of gravity waves away from the region of surface disturbance. The course will not examine gravity waves on the free surface as they are not applicable to many chemical engineering processes.

9.3 Velocity Profiles

As we did in Section 3.5, we can look at the flow patterns within the fluid. The overall velocity is constant, but the velocity at different points in the channel varies. In this case the velocity on the free-surface will be the highest value and near the walls the velocity will be reduced. This means that the velocity is dependant on the height position within the channel.

9.3.1 Laminar Flow

The simplest case for open-channel flow is steady-state and uniform, down a slightly inclined plane. It is possible to derive the velocity profile for a fluid under laminar flow conditions with knowledge of the velocity. Let us take a long, straight, infinite width (i.e. no sides) section of an open-channel which we can think of as a fully developed laminar flow. The velocity profile is the same at any cross section of the channel. Taking a control volume, as in Figure 9.2 and applying a force balance to this we get a balance between gravity and viscous forces.

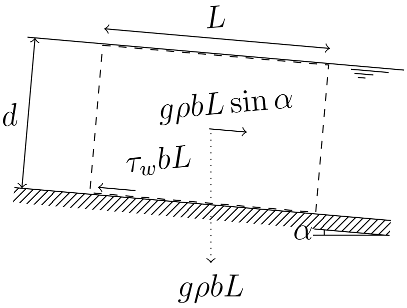


Figure 9.2: Forces on a free body of fluid in steady uniform flow down an inclined plane.

The freely deformable upper surface of the liquid, called the free surface, is open to the air. We will neglect the minor forces exerted by the overlying air on the moving liquid. Take the x-direction to be downstream and the y-direction to be normal to the boundary, with $y = 0$ at the bottom of the flow. By the no-slip condition, the velocity is zero at $y = 0$, so the velocity must increase upward in the flow. It is also clear that the flow is everywhere directed straight down the plane. Think about the forces acting on the fluid contained at a given instant in the free body within the rectangular volume formed by the free surface, the bottom boundary, and two pairs of imaginary planes normal to the bottom and with unit spacing, one pair parallel to the flow and spaced a distance b apart, and the other normal to the flow and spaced a distance L apart.

Balancing the forces on this free body means equating the downslope driving force, caused by the downslope component of the weight of the fluid in the free body, with the resistance force exerted by the planar boundary on the lower surface of the free body. The weight of the fluid in the free body is $g\rho bLd$, where d is the depth of flow. The downslope component of this weight is therefore,

$$F_{w,x} = g\rho bLd \sin\alpha \tag{9.3.1}$$

where α is the slope angle of the plane.

This is balanced by the frictional force exerted by the bottom boundary,

$$F_{f,x} = -\tau_w bL \tag{9.3.2}$$

There are also pressure forces acting parallel to the flow direction on the upstream and downstream faces of the free body, but because by our assumption of uniformity the vertical distribution of these pressure forces is the same at every cross section, they balance each other out and cause no net force on the free body.

The sum of $F_{w,x}$ and $F_{f,x}$ must be equal to zero as they system is steady state, therefore,

$$\begin{aligned} g\rho bLd\sin\alpha - \tau_w bL &= 0 \\ g\rho bLd\sin\alpha &= \tau_w bL \\ \tau_w &= g\rho d\sin\alpha \end{aligned} \tag{9.3.3}$$

so the boundary shear stress is directly proportional to the product of the flow depth, d, the specific weight of the liquid $g\rho$, and the sine of the slope angle, α.

We can find the shear stress and velocity at all points up in the flow by applying the same force-balancing procedure to a free body of fluid similar to that used above but with its lower face formed by an imaginary plane a variable distance y above the bottom and parallel to it (Figure 9.3). The shear stress across the plane is therefore given directly by the force balance,

$$\tau = g\rho\,(d-y)\sin\alpha \tag{9.3.4}$$

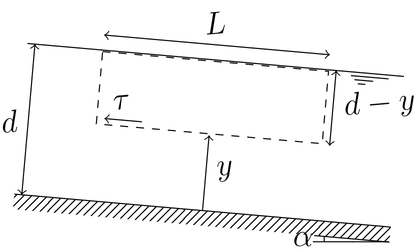


Figure 9.3: Definition sketch for deriving the distribution of shear stress in steady uniform laminar open-channel flow.

To carry the analysis further we must prescribe how the shear stress is related to the velocity of the fluid. As with pipe flow, this is the critical step that separates the analysis of laminar flow from that of turbulent flow, i.e. the shear stress dependence for turbulent flow is very complex and has no current analytical solution. However, for laminar flow of a Newtonian fluid, the shear stress is simply proportional to the velocity gradient as in equation 3.4.5. In the notation of this example this becomes,

$$\tau = \mu\frac{\mathrm{d}\,u}{\mathrm{d}\,y} \tag{9.3.5}$$

so we get, using Equation 9.3.4,

$$\mu \frac{\mathrm{d}\,u}{\mathrm{d}\,y} = g\rho\,(d-y)\sin\alpha$$

$$\frac{\mathrm{d}\,u}{\mathrm{d}\,y} = \frac{g\rho\sin\alpha}{\mu}\,(d-y) \tag{9.3.6}$$

This equation can be integrated to give the velocity distribution from the bottom boundary to the free surface, taking into account that $u = 0$ when $y = 0$ (and that $\tau = 0$ when $y = d$),

$$\int_0^u \mathrm{d}\,u = \int_0^y \frac{g\rho\sin\alpha}{\mu}\,(d-y)\,\mathrm{d}\,y$$

$$[u]_0^u = \frac{g\rho\sin\alpha}{\mu}\left[dy - \frac{y^2}{2}\right]_0^y$$

$$u = \frac{g\rho\sin\alpha}{\mu}\left(dy - \frac{y^2}{2}\right) \tag{9.3.7}$$

For given values of g, ρ, μ, α, and d the velocity u thus varies parabolically from zero at the bottom boundary to a maximum at the surface, Figure 9.4(a).

The volumetric flowrate, Q, through the rectangular channel can be obtained by integrating the velocity profile across the channel, $y = 0$ to $y = d$. In this case, we define a width, b, so long as this is large compared to the depth d then it is a good approximation for the infinitely wide channel.

$$\mathrm{d}\,Q = u\mathrm{d}\,A = ub\mathrm{d}\,y$$

$$\mathrm{d}\,Q = \frac{g\rho\sin\alpha}{\mu}\left(dy - \frac{y^2}{2}\right) b\mathrm{d}\,y$$

$$\int_0^Q \mathrm{d}\,Q = \int_y^d \frac{g\rho\sin\alpha}{\mu}\left(dy - \frac{y^2}{2}\right) b\mathrm{d}\,y$$

$$[Q]_0^Q = \frac{g\rho b\sin\alpha}{\mu}\left[d\frac{y^2}{2} - \frac{y^3}{6}\right]_0^d$$

$$Q = \frac{g\rho b\sin\alpha}{\mu}\left(\frac{d^3}{2} - \frac{d^3}{6}\right)$$

$$Q = \frac{g\rho b d^3\sin\alpha}{3\mu} \tag{9.3.8}$$

By definition, the average velocity is the total flowrate divided by the cross-sectional area of the channel bd, as,

$$\bar{u} = \frac{Q}{A}$$

$$\bar{u} = \frac{\frac{g\rho b d^3\sin\alpha}{3\mu}}{bd} \tag{9.3.9}$$

Laminar Flow Average Velocity

$$\bar{u} = \frac{g\rho d^2 \sin\alpha}{3\mu} \tag{9.3.10}$$

As the maximum velocity is at $y = d$ then,

$$u_{\text{max}} = \frac{g\rho d^2 \sin\alpha}{2\mu} \tag{9.3.11}$$

thus,

$$u_{\text{max}} = \frac{3}{2}\bar{u} \tag{9.3.12}$$

which can be compared with a circular pipe, Equation 3.5.9.

9.3.2 Turbulent Flow

For turbulent flow we have the same issue as with pipe flow in that the link between the shear rate and the velocity gradient is not simple due to the irregularity of the flow in the turbulent regime, i.e. not streamlines. Thus, only the average motion of the fluid is in the direction of flow (parallel to the axis of the channel base). Thus the velocity profile illustrated in Figure 9.4(b) shows the average velocity at different fluid depths.

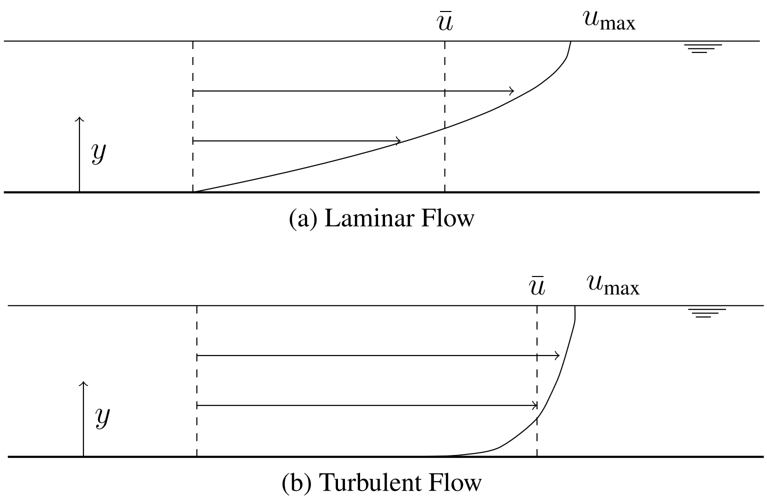


Figure 9.4: Flow profiles in rectangular open channels under (a) laminar and (b) turbulent flow.

9.4 Flow Regimes

As with pipeline flow we still have laminar and turbulent flow discussed above and defined by the same range of Reynolds number Re < 2100 for laminar flow and Re $> 10,000$ for turbulent flow, where the Reynolds number is given by,

$$\text{Re} = \frac{\rho u D_H}{\mu} \tag{9.4.1}$$

where D_H is the hydraulic diameter given by,

$$D_H = \frac{4A_c}{P} \tag{9.4.2}$$

with A_c as the cross-sectional area of the channel and P as the wetted perimeter length (the distance along the submerged part of the boundary from waterline to waterline). The hydraulic radius can also be defined (it is not the same as the ratio of the radius of a circle to the diameter of a circle) as,

$$R_H = \frac{A_c}{P} = \frac{D_H}{4} \tag{9.4.3}$$

Appendix B.6 has some values shown for standard open-channel shapes.

We can also define the flow based on the Froude number,

$$\text{Fr} = \frac{u}{\left(\frac{gA_c}{T}\right)^{0.5}} \tag{9.4.4}$$

where T is the width of the free surface of the flow. The Froude number is the square root of the ratio of the inertial force, mu^2/l, to the gravitational force, mg.

This means for a rectangular channel the Froude number is,

$$\text{Fr}_{\text{rect}} = \frac{u}{(gd)^{0.5}} \tag{9.4.5}$$

and the critical depth can be defined for Fr $= 1$ as,

$$d = \frac{u^2}{g} \tag{9.4.6}$$

The Froude number can also be thougt of as the ratio of the water velocity to the speed of a wave in the shallow water (simiar to the speed of sound in a gas). For a Froude number equal to one the mean flow velocity is equal to the speed of surface waves. A water-surface wave that is moving in the upstream direction (i.e. in the direction opposite to the flow) appears to an observer on the channel bank to be standing still. This means that if the Froude number of the flow is greater than one, wavelike disturbances cannot propagate upstream; the flow coming from upstream cannot know what is in store for it at positions downstream. This is called supercritical flow and the depth is small than the critical depth (i.e. depth at $Fr = 1$). If the Froude number is less than one then this is called subcritical flow, where the upstream flow can be influenced, commonly for long distances, by conditions downstream and the depth is larger than the critical depth.

Figure 9.5 shows these key flow regimes.

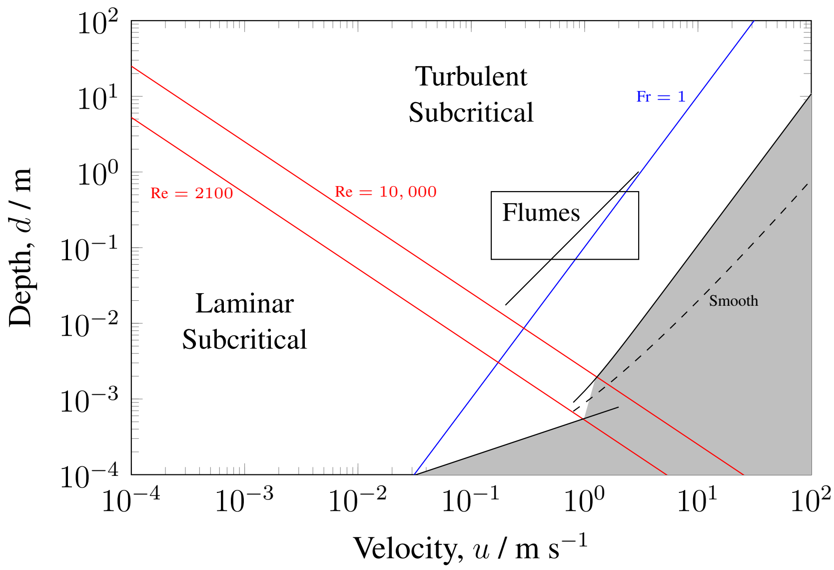


Figure 9.5: Hydraulic regimes of open-channel flow in a graph of mean flow depth vs. mean flow velocity for water in an infinitely wide rectangular channel. The red line is the transition between laminar and turbulent and the blue line is the transition between subcritical and super critical flow. The greyed out area is impossible to achieve with gravity flow (on Earth) as limit lines represent a rough vertical surface, a smooth surface is also shown for turbulent flow. The rectangle shows the best area for man-made open-channel flumes to be designed and the line through this represents a typical 0.2% sloped channel.

9.5 Determination of Flow Height

For Equation 9.3.3, we balanced the fluid downslope weight, Equation 9.3.1, against the downslope friction, Equation 9.3.2. This was undertaken for a rectangular channel but can also be performed for a general shaped channel, such that the fluid downslope weight,

$$F_{w,x} = g\rho L A_c \sin\alpha \tag{9.5.1}$$

where, A_c is the area of the cross-section of the arbitrary cross-section shaped channel. The downslope friction can be given by,

$$F_{f,x} = -\tau_w P L \tag{9.5.2}$$

where P is the wetted perimeter length. Again, the sum of $F_{w,x}$ and $F_{f,x}$ must be equal to zero as the system is steady state, therefore,

$$\begin{aligned} g\rho L A \sin\alpha &= \tau_w P L \\ \tau_w &= g\rho \frac{A_c}{P} \sin\alpha = g\rho R_H \sin\alpha \end{aligned} \tag{9.5.3}$$

where R_H is the hydraulic radius, equation 9.4.3. Though as will become apparent later, it is often more convenient to use the hydraulic diameter, D_H, equation 9.4.2, thus, the wall shear stress can be given by,

$$\tau_w = \frac{g\rho D_H \sin\alpha}{4} \tag{9.5.4}$$

From Equation 6.3.10 we defined,

$$f = \frac{\tau_w}{0.5\rho u^2} \tag{9.5.5}$$

Using this to eliminate the wall shear stress from Equation 9.5.4 gives,

$$f\frac{\rho u^2}{2} = \frac{g\rho D_H \sin\alpha}{4}$$
$$u = \left(\frac{g}{2f}D_H \sin\alpha\right)^{1/2} = \left(\frac{2g}{f}R_H \sin\alpha\right)^{1/2} \tag{9.5.6}$$

We can define the streamwise slope, S, as the meters of drop per meter of horizontal travel, thus for a shallow slope, with small angle, $S = \tan\alpha \approx \sin\alpha \approx \alpha$ such that,

$$u = \left(\frac{g}{2f}D_H S\right)^{1/2} \tag{9.5.7}$$

9.5.1 Chézy Coefficient

The eighteenth-century French hydraulic engineer, Antoine de Chézy [1] adapted the above equation to become,

Chézy Equation

$$u = C\left(\frac{D_H}{4}S\right)^{1/2} \tag{9.5.8}$$

where,

$$C = \left(\frac{2g}{f}\right)^{1/2} \tag{9.5.9}$$

is called the Chézy coefficient. It is introduced here as it is common in the use of work on open-channel flow; however, it is no different to just calculation directly from the friction factor. The coefficient is not a dimensionless number like the friction factor; it has the dimensions $\mathrm{m}^{1/2}\,\mathrm{s}^{-1}$. The friction factor used here is the fanning friction factor, be careful not to confuse this with the Darcy-Weisbach friction factor, λ, which is also sometimes used for the Chézy coefficient and would be given by,

$$C = \left(\frac{8g}{\lambda}\right)^{1/2} \tag{9.5.10}$$

In turbulent flow the friction factor value is identical to that in pipe flow, and thus can be given by the Colebrook equation, equation 6.4.5. In the laminar flow the friction factor can be given by $f = K/\mathrm{Re}$ where K is a constant which depends on the shape of the channel. We have already seen for fully contained circular pipe flow that $K = 16$, equation 6.4.2 and comparing equation 9.5.8 and 9.5.9 with equation 9.3.10 we get $K = 24$ for an infinitely wide rectangular channel. Values have been determined experimentally for other rectangular aspect ratios as shown in Figure 9.6 [3]. Triangular cross-section channels with lower values, $K = 14$.

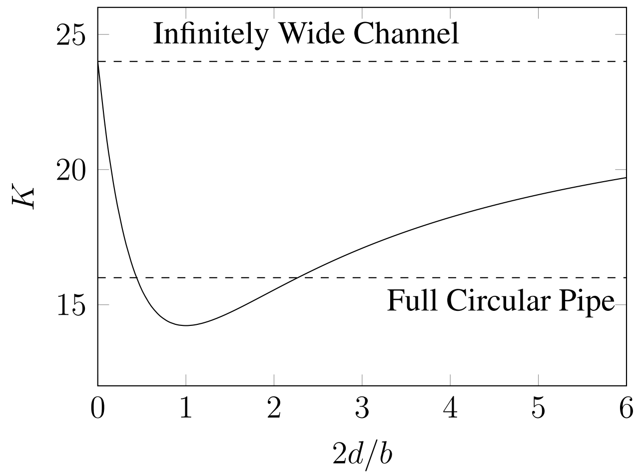


Figure 9.6: Variation of K with rectangular open-channel aspect ratio. Infinitely wide channel is $2d/b = 0$.

9.5.2 Manning Coefficient

In 1890, Irish engineer Robert Manning reworked an empirical equation originally developed by the French engineer Philippe Gauckler in 1867 which approximated the Chézy coefficient for different channel materials when the flowing fluid was water in the turbulent regime as [2],

Manning Coefficient

$$C = \frac{1}{n} R_H^{1/6} = \frac{1}{4^{1/6} n} D_H^{1/6} \tag{9.5.11}$$

The Manning roughness coefficient, n, is a dimensional empirical constant, its value depends on the nature of the channel and its surfaces, Appendix B.5 provides a list of coefficients.

9.5.3 Compound Channels

Combining Manning's formula with the formula for the flow rate gives,

$$\begin{aligned} Q &= A_c u \\ &= A_c \left(\frac{1}{n} R_H^{1/6} \right) (R_H S)^{1/2} \\ &= A_c \frac{1}{n} R_H^{2/3} S^{1/2} \\ &= \frac{1}{n} \frac{A_c^{5/3}}{P^{2/3}} S^{1/2} \\ &= K S^{1/2} \end{aligned} \tag{9.5.12}$$

where K is often called the conveyance of the channel. The primary use of K is in determining the discharge capacity of compound channels – for example a main channel and overflow sections. By adding the contribution to total discharge from individual components with different roughness,

$$Q_T = \sum_i Q_i = \sum_i K_i S^{1/2} = K_{\text{eff}} S^{1/2} \tag{9.5.13}$$

Therefore, assuming each part of the channel has the same slope then,

$$K_{\text{eff}} = \sum_i \frac{1}{n_i} \frac{A_{c,i}^{5/3}}{P_i^{2/3}} \tag{9.5.14}$$

It should be noted here that the perimeter, P_i, is the wetted perimeter, thus the side where the water in the channels meet should not be included.

9.6 Bernoulli's Equation for Open-Channels

As discussed in Chapter 4, Bernoulli's equation is an expression of the work–energy theorem; the work done by the fluid pressure is equal to the change in kinetic and potential energies of the flow,

$$\frac{P_1}{\rho g} + \frac{u_1^2}{2g} + h_1 = \frac{P_2}{\rho g} + \frac{u_2^2}{2g} + h_2 + \Delta\, h_f$$

In this case, the pressure on the flow surface is always atmospheric pressure regardless of the height of the channel, therefore the pressure in the open-channel can be given by $P = P_{\text{atm}} + \rho g d$ so,

Open Channel Bernoulli's Equation

$$d_1 + \frac{u_1^2}{2g} + h_{o,1} = d_2 + \frac{u_2^2}{2g} + h_{o,2} + \Delta\, h_f \tag{9.6.1}$$

where h_o is the height of the base of the channel[1] and the head loss due to friction can be taken from equation 6.3.11 and modifided for a general channel shape as,

$$\Delta\, h_f = f \frac{4L}{D_H} \frac{u^2}{2g} \tag{9.6.2}$$

In advanced analysis it is necessary to precede the kinetic energy term $u^2/2g$ by a corrective multiplicative factor α (the kinetic energy correction coefficient) to account for the fact that the velocity profile is not uniform, and hence the mean squared velocity $< u^2 >$ is not equal to the square of the mean velocity $< u >^2$. For fully turbulent flow in large channels, α is typically about 1.02; i.e. very close to 1. Hence, this factor will be ignored here.

For a steady slope channel, for equation 9.5.7 we defined the streamwise slope, S, as the meters of drop per meter of horizontal travel, we can also define a friction slope in a similar manner as,

$$S_f = \frac{\Delta\, h_f}{L} = f \frac{4}{D_H} \frac{u^2}{2g} \tag{9.6.3}$$

[1]The height of the fluid at any position in the channel can be given by $h_o + y$ and the gauge pressure can be given by $\rho g\,(d - y)$, thus substituting this into the sum of the potential and pressue heads gives,

$$h + \frac{P}{\rho g} = h_o + y + \frac{\rho g\,(d-y)}{\rho g} = h_o + y + d - y = h_o + d$$

Thus for a steady slope open-channel Bernoulli's equation, equation 9.6.1, is sometimes modified to be,

$$d_1 + \frac{u_1^2}{2g} = d_2 + \frac{u_2^2}{2g} + (S_f - S)\, L \tag{9.6.4}$$

In uniform flow, $d_1 = d_2$ and $u_1 = u_2$ thus we get $S = S_f$ which is equivalent to equation 9.5.7.

9.7 References

[1] Chanson, H. [2004], *Hydraulics of Open Channel Flow*, Elsevier.

[2] Manning, R. [1891], 'On the flow of water in open channels and pipes.', *Transactions of the Institute of Civil Engineers of Ireland* **20**, 161–209.

[3] Tsanis, I. K. and Leutheusser, H. J. [1986], 'Hydraulics of laminar open-channel flow', *Journal of Hydraulic Research* **24**, 193–206.

Appendix A

Vector Calculus

A.1 Vector Fields

Mathematically a vector field is defined as,

$$\mathbf{F} : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \tag{A.1.1}$$

which means that a vector field assigns to each point $\mathbf{x}$ in X a vector $\mathbf{F}(\mathbf{x})$ in $\mathbb{R}^n$, represented by an arrow whose tail is at point $\mathbf{x}$.

Taking this to be an example that we understand in 3-dimensions would be a velocity vector applied to co-ordinates in space,

$$\mathbf{V}(x, y, z) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \tag{A.1.2}$$

A single 3D vector is visualised as Figure A.1(a). For the vector field, each point in space then has a velocity vector applied to it, the values of u, v, and w can be functions of the position of the vector as in Figure A.1(b).

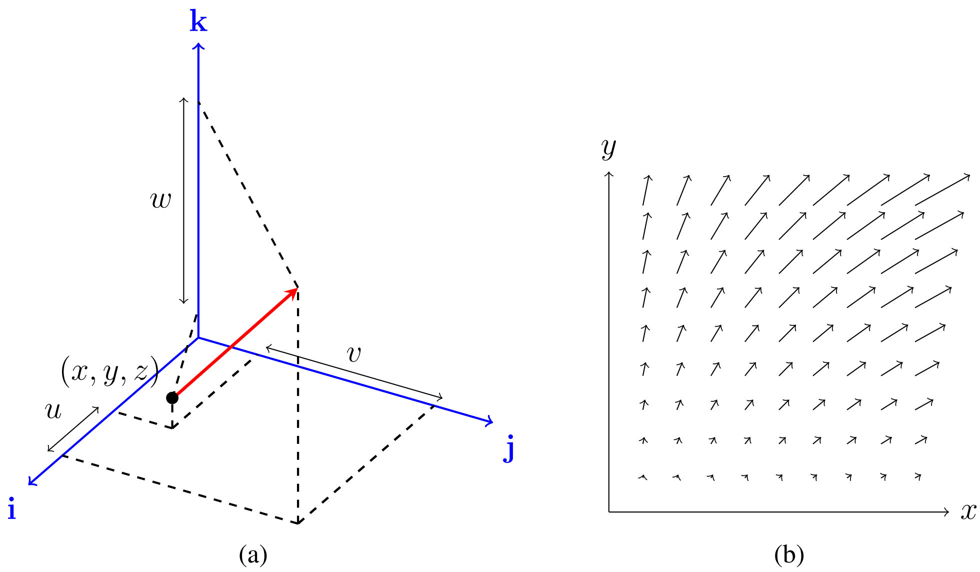


Figure A.1: Velocity vectors, (a) A 3D vector $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ at a point $X(x, y, z)$, and (b) a 2D vector field where $\mathbf{V}(x, y) = 0.2xy\mathbf{i} + 0.1y\mathbf{j}$.

The magnitude of the vector at each position is then given by,

$$\|\mathbf{V}\| = \sqrt{u^2 + v^2 + w^2} \tag{A.1.3}$$

A.2 Differentiation

A.2.1 Gradient Fields

The gradient of a scalar-valued differentiable function f of several variables is the vector field whose value at a point is the vector whose components are the partial derivatives of f at that point, such that,

$$\text{If } \mathbf{f} : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$$
$$\text{Then } \nabla\mathbf{f} : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$$

As an example, consider a room where the temperature is given by a scalar field, T, so at each point (x, y, z) the temperature is $T(x, y, z)$, independent of time. At each point in the room, the gradient of T at that point will show the direction in which the temperature rises most quickly, moving away from (x, y, z). The magnitude of the gradient will determine how fast the temperature rises in that direction.

Mathematically, this can be written as,

$$\nabla \mathbf{f} = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{\partial f}{\partial x_2}\mathbf{e}_2 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n \tag{A.2.1}$$

where $x_1, x_2, \cdots, x_n$ are the coordinates for $\mathbb{R}^n$ and $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ are the unit vectors in the vector field space of $\mathbb{R}^n$.

Taking an example of a Pressure distribution in 3-dimensions $\mathbf{P}(x, y, z)$ where $\mathbf{P}$ is a function of x, y, and z and calculating the gradient gives,

$$\begin{aligned} \mathbf{P} &= x^2yz \\ \nabla \mathbf{P} &= \frac{\partial x^2yz}{\partial x}\mathbf{i} + \frac{\partial x^2yz}{\partial y}\mathbf{j} + \frac{\partial x^2yz}{\partial z}\mathbf{k} \\ &= 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k} \end{aligned}$$

It is possible to have a gradient of a vector field and this produces a tensor such as,

$$\nabla \mathbf{F} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \tag{A.2.2}$$

A.2.2 Divergence of a Vector Field

The divergence is a vector operator that operates on a vector field, producing a scalar field giving a measure of the vector field's source at each point. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.

As an example, consider air as it is heated or cooled. The velocity of the air at each point defines a vector field. While air is heated in a region, it expands in all directions, and thus the velocity field points outward from that region. The divergence of the velocity field in that region would thus have a positive value. While the air that is cooled and thus contracting, the divergence of the velocity in that region has a negative value.

Mathematically, this can be written as,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n} \tag{A.2.3}$$

where $x_1, x_2, \cdots, x_n$ are the coordinates for $\mathbb{R}^n$ and $F_1, F_2, \cdots, F_n$ are the components of the vector field $\mathbf{F}$.

Taking an example velocity vector field in 3-dimensions $\mathbf{V}(x, y, z) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ where u, v, and w are functions of x, y, and z and calculating the divergence gives,

$$\begin{aligned}\mathbf{V} &= x^2y\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k}\\ \nabla \cdot \mathbf{V} &= \frac{\partial\, x^2y}{\partial\, x} + \frac{\partial\, xz}{\partial\, y} + \frac{\partial\, xyz}{\partial\, z}\\ &= 2xy + 0 + xy = 3xy\end{aligned}$$

A.2.3 Curl of a Vector Field

The curl is a vector operator that operates on a vector field and that describes the circulation of a vector field. The curl at a point in the field is represented by a vector whose length and direction denote the magnitude and axis of the maximum circulation, it only operates on a 3-dimensional vector field on a 3-dimensional coordinate system, $\mathbf{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$.

As an example, suppose the vector field describes the velocity field of a fluid flow (such as a large tank of liquid) and a small ball is located within the fluid (the centre of the ball being fixed at a certain point). If the ball has a rough surface, the fluid flowing past it will make it rotate. The rotation axis points in the direction of the curl of the field at the centre of the ball, and the angular speed of the rotation is half the magnitude of the curl at this point.

Mathematically, this can be written as,

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial\, x_1} & \frac{\partial}{\partial\, x_2} & \frac{\partial}{\partial\, x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}\\ &= \left(\frac{\partial\, F_3}{\partial\, x_2} - \frac{\partial\, F_2}{\partial\, x_3}\right)\mathbf{i} + \left(\frac{\partial\, F_1}{\partial\, x_3} - \frac{\partial\, F_3}{\partial\, x_1}\right)\mathbf{j} + \left(\frac{\partial\, F_2}{\partial\, x_1} - \frac{\partial\, F_1}{\partial\, x_2}\right)\mathbf{k} \qquad \text{(A.2.4)}\end{aligned}$$

where x_1, x_2, x_3 are the coordinates for $\mathbb{R}^3$ and F_1, F_2, F_3 are the components of the vector field $\mathbf{F}$.

Taking an example velocity vector field in 3-dimensions $\mathbf{V}(x, y, z) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ where u, v, and w are functions of x, y, and z and calculating the curl gives,

$$\begin{aligned}\mathbf{V} &= x^2y\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k}\\ \nabla \times \mathbf{V} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial\, x} & \frac{\partial}{\partial\, y} & \frac{\partial}{\partial\, z} \\ x^2y & xz & xyz \end{vmatrix}\\ &= \left(\frac{\partial\, xyz}{\partial\, y} - \frac{\partial\, xz}{\partial\, z}\right)\mathbf{i} + \left(\frac{\partial\, x^2y}{\partial\, z} - \frac{\partial\, xyz}{\partial\, x}\right)\mathbf{j} + \left(\frac{\partial\, xz}{\partial\, x} - \frac{\partial\, x^2y}{\partial\, y}\right)\mathbf{k}\\ &= (xz - x)\,\mathbf{i} + (0 - yz)\,\mathbf{j} + \left(z - x^2\right)\mathbf{k}\end{aligned}$$

A.2.4 Substantial Derivative

The substantial derivative, or material derivative, describes the time rate of change of some physical quantity of a material element that is subjected to a space-and-time-dependent macroscopic velocity field.

This means that our physical quantity, in 3-dimensions, is therefore,

$$\varphi \to \varphi\,(t, x(t), y(t), z(t)) \tag{A.2.5}$$

The expression $\Delta\,\varphi$ is the change in φ when comparing the value of the function φ at two nearby points, (t, x, y, z) and $(t + \Delta\,t, x + \Delta\,x, y + \Delta\,y, z + \Delta\,z)$. This change can be given by the difference between the 2 points times by the change in φ in that relevant direction, such as,

$$\Delta\,\varphi = \frac{\partial\,\varphi}{\partial\,t}\Delta\,t + \frac{\partial\,\varphi}{\partial\,x}\Delta\,x + \frac{\partial\,\varphi}{\partial\,y}\Delta\,y + \frac{\partial\,\varphi}{\partial\,z}\Delta\,z \tag{A.2.6}$$

In fluid mechanics there is a particular path and set of neighbouring particles that is of interest; and that is the path that fluid particles take. If we choose one such piece of fluid, its motion describes a path through three-dimensional space, a streamline. As the fluid particles are in a particular path, then we can relate the change in direction; $\Delta\,x$, $\Delta\,y$, and $\Delta\,z$; to the local fluid velocity components; u, v, and w, such as,

$$\text{Along a streamline: } \begin{cases} \Delta\,x = u\Delta\,t \\ \Delta\,y = v\Delta\,t \\ \Delta\,z = w\Delta\,t \end{cases} \tag{A.2.7}$$

This means that,

$$\begin{aligned} \Delta\,\varphi &= \frac{\partial\,\varphi}{\partial\,t}\Delta\,t + \frac{\partial\,\varphi}{\partial\,x}u\Delta\,t + \frac{\partial\,\varphi}{\partial\,y}v\Delta\,t + \frac{\partial\,\varphi}{\partial\,z}w\Delta\,t \\ \left.\frac{\Delta\,\varphi}{\Delta\,t}\right|_{\text{Streamline}} &= \frac{\partial\,\varphi}{\partial\,t} + \frac{\partial\,\varphi}{\partial\,x}u + \frac{\partial\,\varphi}{\partial\,y}v + \frac{\partial\,\varphi}{\partial\,z}w \end{aligned} \tag{A.2.8}$$

To achieve the true derivative we can take the limit as $\Delta\,t$ goes to 0, which can be defined as the substantial derivative,

$$\begin{aligned} \frac{D\,\varphi}{D\,t} &= \lim_{\Delta\,t \to 0} \left.\frac{\Delta\,\varphi}{\Delta\,t}\right|_{\text{Streamline}} \\ &= \frac{\partial\,\varphi}{\partial\,t} + \frac{\partial\,\varphi}{\partial\,x}u + \frac{\partial\,\varphi}{\partial\,y}v + \frac{\partial\,\varphi}{\partial\,z}w \end{aligned} \tag{A.2.9}$$

From knowledge of the gradient (Section A.2.1) and taking a velocity vector field, we can see that this can also be written as,

$$\frac{D\,\varphi}{D\,t} = \frac{\partial\,\varphi}{\partial\,t} + (\mathbf{V} \cdot \nabla)\,\varphi \tag{A.2.10}$$

The substantial derivative contains two terms, the first $\partial\,\varphi/\partial\,t$ term is called the local derivative, the second $\mathbf{V} \cdot \nabla\varphi$ term is called the convective derivative.

A.2.5 Vector Identities

The below are some useful vector calculus identities, they are not an exhaustive list as an they are only for reference they will not be proved here,

$$
\begin{aligned}
\nabla(\varphi+\phi) &= \nabla\varphi+\nabla\phi \\
\nabla(\mathbf{A}+\mathbf{B}) &= \nabla\mathbf{A}+\nabla\mathbf{B} \\
\nabla\cdot(\mathbf{A}+\mathbf{B}) &= \nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B} \\
\nabla\times(\mathbf{A}+\mathbf{B}) &= \nabla\times\mathbf{A}+\nabla\times\mathbf{B} \\
\nabla(\varphi\phi) &= \phi\nabla\varphi+\varphi\nabla\phi \\
\nabla(\varphi\mathbf{A}) &= (\nabla\varphi)^T\mathbf{A}+\varphi\nabla\mathbf{A} \\
\nabla\cdot(\varphi\mathbf{A}) &= \varphi\nabla\cdot\mathbf{A}+(\nabla\varphi)\cdot\mathbf{A} \\
\nabla\times(\varphi\mathbf{A}) &= \varphi\nabla\times\mathbf{A}+(\nabla\varphi)\times\mathbf{A} \\
\nabla(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}) \\
&= (\nabla\mathbf{B})\cdot\mathbf{A}+(\nabla\mathbf{A})\cdot\mathbf{B} \\
\nabla(\mathbf{A}\cdot\mathbf{A}) &= 2(\mathbf{A}\cdot\nabla)\mathbf{A}+2\mathbf{A}\times(\nabla\times\mathbf{A}) = 2(\nabla\mathbf{A})\cdot\mathbf{A} \\
(\mathbf{A}\cdot\nabla)\mathbf{A} &= \frac{1}{2}\nabla|\mathbf{A}|^2-\mathbf{A}\times(\nabla\times\mathbf{A}) \\
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A} \\
\nabla\cdot(\mathbf{A}\otimes\mathbf{B}) &= (\nabla\cdot\mathbf{A})\mathbf{B}+\mathbf{A}\cdot\nabla\mathbf{B}
\end{aligned}
$$

A.3 Integration

A.3.1 Surface Integral

A surface integral is a generalization of multiple integrals to integration over a particular surface. Given a surface, one may integrate a scalar field over the surface, or a vector field over the surface.

In this case we are interested in the surface intergration with a vector field. Consider a vector field **F** on a surface Γ so that for each position on the surface $\mathbf{p} = (x, y, z) \to \Gamma$, $\mathbf{F}(p)$ is a vector.

If the vector field is tangent to Γ at each point, then the flux passing through the surface is zero because the fluid just flows in parallel to Γ, and neither in nor out. If the vector field is normal to Γ at each point, then the flux passing through the surface is equal to the vector field as it all flows over the surface. These imply that if **F** does not just flow along or through Γ, that is, if **F** has both a tangential and a normal component, then only the normal component contributes to the flux. Based on this reasoning, to find the flux, we need to take the dot product of **F** with the unit surface, which gives,

$$\oiint_\Gamma \mathbf{F}\cdot\mathrm{d}\mathbf{\Gamma} = \oiint_\Gamma (\mathbf{F}\cdot\mathbf{n})\,\mathrm{d}\Gamma \tag{A.3.1}$$

where n is the outward-pointing normal vector at the relevent point to Γ, thus taking only the normal flow component of **F** as in Figure A.2.

If our vector field is taken to be the velocity then we actually get,

$$\oiint_\Gamma (\mathbf{V}\cdot\mathbf{n})\,\mathrm{d}\Gamma = uA = Q$$

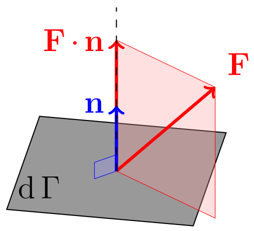


Figure A.2: Vector field passing through a surface element.

i.e. the volumetric flow rate.

A.3.2 Volume Integral

A volume intergral is an integral over a 3-dimensional domain, Ω, enclosed by a closed surface Γ,

$$\oiiint_{\Omega} \mathbf{F} \, \mathrm{d}\,\Omega \tag{A.3.2}$$

This can either be taken on a scalar or vector field.

If our scalar field is taken to be the density then we actually get,

$$\oiiint_{\Omega} \rho \mathrm{d}\,\Omega = \rho V = m$$

i.e. the total mass of our object.

A.3.3 Divergence Theorem

The divergence theorem, also known as Gauss's theorem, is a theorem which relates the flux of a vector field through a closed surface to the divergence of the field in the volume enclosed. More precisely, the divergence theorem states that the surface integral of a vector field over a closed surface, which is called the flux through the surface, is equal to the volume integral of the divergence over the region inside the surface.

The divergence theorem is typically applied in three dimensions; However, it generalizes to any number of dimensions (In one dimension, it is equivalent to integration by parts; in two dimensions, it is equivalent to Green's theorem).

Vector fields are often illustrated using the example of the velocity field of a fluid, such as a gas or liquid. A moving liquid has a velocity—a speed and a direction—at each point, which can be represented by a vector, so that the velocity of the liquid at any moment forms a vector field. Consider an imaginary closed surface S inside a body of liquid, enclosing a volume of liquid. The flux of liquid out of the volume is equal to the volume rate of fluid crossing this surface, i.e., the surface integral of the velocity over the surface.

Since liquids are incompressible, the amount of liquid inside a closed volume is constant; if there are no sources or sinks inside the volume then the flux of liquid out of S is zero. If the liquid is moving, it may flow into the volume at some points on the surface S and out of the volume at other points, but the amounts flowing in and out at any moment are equal, so the net flux of liquid out of the volume is zero.

However if a source of liquid is inside the closed surface, such as a pipe through which liquid is introduced, the additional liquid will exert pressure on the surrounding liquid, causing an outward flow in all directions. This will cause a net outward flow through the surface S. The flux outward through S equals the volume rate of flow of fluid into S from the pipe. Similarly if there is a sink or drain inside S, such as a pipe which drains the liquid off, the external pressure of the liquid will cause a velocity throughout the liquid directed inward toward the location of the drain. The volume rate of flow of liquid inward through the surface S equals the rate of liquid removed by the sink.

If there are multiple sources and sinks of liquid inside S, the flux through the surface can be calculated by adding up the volume rate of liquid added by the sources and subtracting the rate of liquid drained off by the sinks. The volume rate of flow of liquid through a source or sink (with the flow through a sink given a negative sign) is equal to the divergence of the velocity field at the pipe mouth, so adding up (integrating) the divergence of the liquid throughout the volume enclosed by S equals the volume rate of flux through S. This is the divergence theorem.

Mathematically, this can be written as,

$$\oiiint_{\Omega} \nabla \cdot \mathbf{F} \,\mathrm{d}\,\Omega = \oiint_{\Gamma} \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}\,\Gamma \tag{A.3.3}$$

where $\Omega \subseteq \mathbb{R}^3$ and $\partial\,\Omega = \Gamma$. This can be mathematically proven, but is beyond the scope of Chemical Engineering.

A.3.4 Reynolds Transport Theorem

the Reynolds transport theorem (also known as the Leibniz–Reynolds transport theorem), or simply the Reynolds theorem, named after Osborne Reynolds (1842–1912), is a three-dimensional generalization of the Leibniz integral rule. It is used to recast time derivatives of integrated quantities and is useful in formulating the basic equations of continuum mechanics.

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \oiiint_{\Omega} \varphi \mathrm{d}\,\Omega = \oiiint_{\Omega} \frac{\partial\,\varphi}{\partial\,t} \mathrm{d}\,\Omega + \oiiint_{\partial\Omega(t)} \left(\mathbf{v}_{\Omega} \cdot \mathbf{n}\right) \varphi \mathrm{d}\,\Gamma \tag{A.3.4}$$

where $\mathbf{v}_{\Omega}$ is the velocity of the element. in which n(x,t) is the outward-pointing unit normal vector, x is a point in the region and is the variable of integration, dV and dA are volume and surface elements at x, and vb(x,t) is the velocity of the area element (not the flow velocity). The function f may be tensor-, vector- or scalar-valued.[4] Note that the integral on the left hand side is a function solely of time, and so the total derivative has been used.

If the volume, Ω, is constant with respect to time then $\mathbf{v}_{\Omega} = 0$ as is the case here.

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \oiiint_{\Omega} \varphi \mathrm{d}\,\Omega = \oiiint_{\Omega} \frac{\partial\,\varphi}{\partial\,t} \mathrm{d}\,\Omega \tag{A.3.5}$$

Appendix B

Standard Pipes and Fittings

B.1 Mild Steel Standard Pipe Sizes

Nominal Bore / inches	Outside Diameter / mm	Pipe Schedule, wall thinkness (t_w) / mm													
		SCH 5	SCH 10	SCH 20	SCH 30	SCH 40	STD	SCH 60	SCH 80	XS	SCH 100	SCH 120	SCH 140	SCH 160	XXS
1/8	10.30					1.73	1.73		2.41	2.41					
1/4	13.70	1.20	1.72			2.24	2.24		3.02	3.02					
3/8	17.10	1.20	1.72			2.31	2.31		3.20	3.20					
1/2	21.30	1.72	2.11			2.77	2.77		3.73	3.73				4.78	7.47
3/4	26.70	1.72	2.11		2.41	2.87	2.87		3.91	3.91		4.32		5.54	7.82
1	33.40	1.72	2.77		2.90	3.38	3.38		4.55	4.55		5.08		6.35	9.09
1 1/4	42.20	1.72	2.77			3.56	3.56		4.85	4.85				6.35	9.70
1 1/2	48.26	1.72	2.77		3.18	3.68	3.68		5.08	5.08		5.72		7.14	10.16
2	60.32	1.72	2.77			3.91	3.91		5.54	5.54		6.35		8.74	11.07
2 1/2	73.02	2.11	3.04			5.16	5.16		7.01	7.01		7.62		9.52	14.02
3	88.90	2.11	3.04			5.49	5.49		7.62	7.62		8.89		11.13	15.24
3 1/2	101.60	2.11	3.04			5.70	5.70		8.10	8.10					15.91
4	114.30	2.11	3.04			6.02	6.02	7.10	8.56	8.56		11.13		13.49	17.12
5	141.30	2.77	3.38			6.55	6.55		9.52	9.52		12.70		15.88	19.10
6	168.27	2.77	3.38			7.11	7.11		10.97	10.97		14.28		18.26	21.95
8	219.07	2.77	3.73	6.35	7.04	8.18	8.18	10.31	12.70	12.70	15.08	18.26	20.63	23.01	22.22
10	273.05	3.38	4.08	6.35	7.80	9.27	9.27	12.70	15.08	12.70	18.26	21.43	25.40	28.58	25.40
12	323.85	3.96	4.55	6.35	8.38	10.31	9.52	14.28	17.48	12.70	21.43	25.40	28.58	32.00	25.40
14	355.60	3.96	6.35	7.93	9.52	11.07	9.52	15.08	19.05	12.70	23.80	27.00	31.75	35.71	
16	406.40		6.35	7.93	9.52	12.70	9.52	16.66	21.40	12.70	26.19	30.95	36.51	40.48	
18	457.20		6.35	7.93	11.07	14.28	9.52	19.10	23.80	12.70	29.30	34.92	39.60	45.24	
20	508.00		6.35	9.52	12.70	15.08	9.52	20.62	26.19	12.70	32.51	38.10	44.50	50.00	
24	609.60		6.35	9.52	14.28	17.48	9.52	24.60	30.95	12.70	38.89	46.03	52.38	59.53	
30	762.00	6.35	8.00	12.70	15.88		9.52			12.70					
36	914.40		8.00	12.70	15.88	19.10	9.52			12.70					

B.2 Copper Standard Drawn Tube Sizes

Nominal Size / inches	Outside Diameter / mm	Pipe Type, wall thinkness (t_w) / mm		
		X	Y	Z
1/8	6	0.6	0.8	0.5
1/4	8	0.6	0.8	0.5
3/8	10	0.6	0.8	0.5
1/2	15	0.7	1	0.5
5/8	18	0.8	1	0.6
3/4	22	0.9	1.2	0.6
1	28	0.9	1.2	0.6
1 1/4	35	1.2	1.5	0.7
1 1/2	42	1.2	1.5	0.8
2	54	1.2	2	0.9
2 1/2	67	1.2	2	1
3	76.2	1.5	2	1.2
4	108.1	1.5	2.5	1.2
5	133.4	1.5		1.5
6	159.4	2		1.5

Type X is the most common and is used in above-ground service, including drinking water supply, hot and cold water systems, sanitation, central heating, and other general purpose applications.

Type Y is a thicker walled pipe, used for underground works and heavy duty requirements, including hot and cold water supply, gas reticulation, sanitary plumbing, heating and general engineering.

Type Z is a thinner walled pipe, also used for above-ground service, including drinking water supply, hot and cold water systems, sanitation, central heating and other general purpose applications.

B.3 Typical Pipe Material Roughness Values

The table below gives some example pipe material roughness values. These are example values and may differ from other sources.

Material	ε / mm
Plastic (and glass)	0.0015
Drawn tubing (brass, lead, copper)	0.0015
Stainless Steel	0.03
Commercial steel (and wrought iron)	0.045
Mild steel (new)	0.04
Mild steel (slightly corroded)	0.08
Mild steel (moderately corroded)	0.5
Mild steel (badly corroded)	2.0
Asphalted Cast iron	0.5
Galvanized iron	0.15
Cast iron	0.25
Wood stave	0.18-0.9
Concrete (smooth)	0.3
Concrete (form marks)	1.0
Riveted Steel	0.9-9.0
Water mains	1.2
Brickwork (mature foul sewers)	3

B.4 Minor Friction Losses in Pipe Fittings

The table below is an example sent of minor friction loss values, K, for different fittings used in,

$$\Delta h_l = K \frac{u^2}{2g} \tag{B.4.1}$$

These are example values and other sources may differ.

Component	K
Regular 90° elbow, flanged	0.3
Regular 90° elbow, threaded	1.5
Long radius 90° elbow, flanged	0.2
Long radius 90° elbow, threaded	0.7
Regular 45° elbow, flanged	0.1
Regular 45° elbow, threaded	0.4
180° return bend, flanged	0.2
180° return bend, threaded	1.5
Through-flow tee, flanged	0.2
Through-flow tee, threaded	0.9
Branch-flow tee, flanged	1.0
Branch-flow tee, threaded	2.0
Union, flanged	0.05
Union, threaded	0.08
Sudden Contraction ($D_s/D_l = 0.50$)	0.32
Sudden Contraction ($D_s/D_l = 0.25$)	0.39
Sudden Contraction ($D_s/D_l = 0.10$)	0.46
Sudden Entrance	0.50
Sudden Expansion ($D_s/D_l = 0.50$)	0.56
Sudden Expansion ($D_s/D_l = 0.25$)	0.88
Sudden Expansion ($D_s/D_l = 0.10$)	0.99
Sudden Exit	1.00
Globe valve, fully open	10
Gate valve, fully open	0.15
Gate valve, $1/4$ closed	0.26
Gate valve, $1/2$ closed	2.1
Gate valve, $3/4$ closed	17
Swing check valve, forward flow	2
Swing check valve, backward flow	∞
Ball valve, fully open	0.05
Ball valve, $1/3$ closed	5.5
Ball valve, $2/3$ closed	210

B.5 Manning Coefficients

The table below is an example sent Manning coefficients, n, as used in,

$$C = \frac{1}{n} R_H^{1/6} \tag{B.5.1}$$

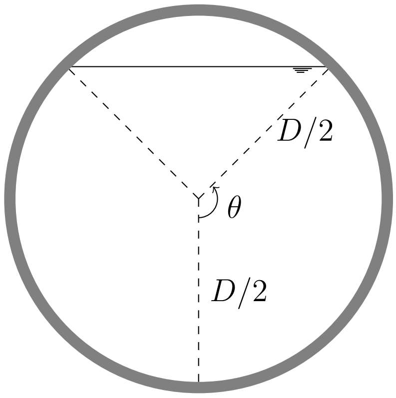
These are example values and other sources may differ.

Material	n / s m$^{-1/3}$
Glass	0.010
Brass, smooth	0.009
Stainless steel	0.013
Steel, riveted	0.015
Mild steel, brushed	0.010
Cast iron	0.013
Wood	0.011
Concrete	0.015
Brickwork	0.015
Asphalt	0.016
Excavated earth, clean	0.022
Excavated earth, gravelly	0.025

B.6 Open-Channel Shapes

The example open-channels below have the relevant cross-sectional areas, A_c, and the wetted perimeter, P, which does not include the free surface.

B.6.1 Circular Channel

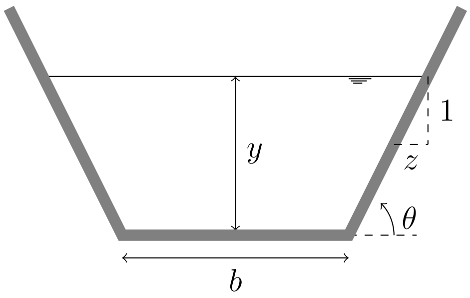


$$A_c = \frac{D^2}{4}\left(\theta - \sin\theta\cos\theta\right) \tag{B.6.1}$$

$$P = D\theta \tag{B.6.2}$$

$$T = D\sin\theta \tag{B.6.3}$$

B.6.2 Symmetric Trapezoidal Channel

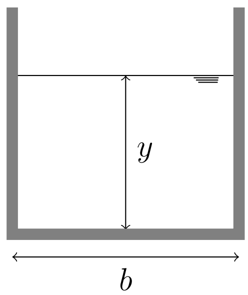


$$A_c = y\left(b + \frac{y}{\tan\theta}\right) = y\,(b + yz) \tag{B.6.4}$$

$$P = b + \frac{2y}{\sin\theta} = b + 2y\left(1 + z^2\right)^{1/2} \tag{B.6.5}$$

$$T = b + \frac{2y}{\tan\theta} = b + 2zy \tag{B.6.6}$$

B.6.3 Rectangular Channel

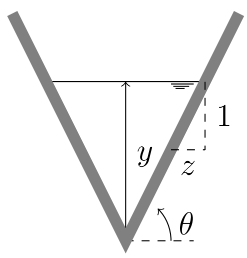


$$A_c = yb \tag{B.6.7}$$

$$P = b + 2y \tag{B.6.8}$$

$$T = b \tag{B.6.9}$$

B.6.4 Symmetric Triangular Channel



$$A_c = \frac{y^2}{\tan\theta} = zy^2 \tag{B.6.10}$$

$$P = \frac{2y}{\sin\theta} = 2y\left(1 + z^2\right)^{1/2} \tag{B.6.11}$$

$$T = \frac{2y}{\tan\theta} = 2zy \tag{B.6.12}$$

Appendix C

Saturated Steam Tables

C.1 Sorted by Temperature

T °C	P kPa	v_g $m^3\,kg^{-1}$	h_f $kJ\,kg^{-1}$	h_g $kJ\,kg^{-1}$	h_{fg} $kJ\,kg^{-1}$	s_f $kJ\,kg^{-1}\,K^{-1}$	s_g $kJ\,kg^{-1}\,K^{-1}$	s_{fg} $kJ\,kg^{-1}\,K^{-1}$
0.01	0.6117	205.991	0.00	2500.9	2500.9	0.0000	9.156	9.156
1	0.6571	192.439	4.18	2502.7	2498.6	0.0153	9.129	9.114
2	0.7060	179.758	8.39	2504.6	2496.2	0.0306	9.103	9.072
3	0.7581	168.008	12.60	2506.4	2493.8	0.0459	9.077	9.031
4	0.8135	157.116	16.81	2508.2	2491.4	0.0611	9.051	8.989
5	0.8726	147.011	21.02	2510.1	2489.0	0.0763	9.025	8.949
6	0.9354	137.633	25.22	2511.9	2486.7	0.0913	8.999	8.908
7	1.0021	128.923	29.43	2513.7	2484.3	0.1064	8.974	8.868
8	1.0730	120.829	33.63	2515.6	2481.9	0.1213	8.949	8.828
9	1.1483	113.304	37.82	2517.4	2479.6	0.1362	8.924	8.788
10	1.2282	106.303	42.02	2519.2	2477.2	0.1511	8.900	8.749
11	1.3130	99.787	46.22	2521.0	2474.8	0.1659	8.875	8.710
12	1.4028	93.719	50.41	2522.9	2472.5	0.1806	8.851	8.671
13	1.4981	88.064	54.60	2524.7	2470.1	0.1953	8.827	8.632
14	1.5990	82.793	58.79	2526.5	2467.7	0.2099	8.804	8.594
15	1.7058	77.875	62.98	2528.3	2465.4	0.2245	8.780	8.556
16	1.8188	73.286	67.17	2530.2	2463.0	0.2390	8.757	8.518
17	1.9384	69.001	71.36	2532.0	2460.6	0.2534	8.734	8.481
18	2.0647	64.998	75.54	2533.8	2458.3	0.2678	8.711	8.443
19	2.1983	61.256	79.73	2535.6	2455.9	0.2822	8.688	8.406
20	2.3393	57.757	83.91	2537.4	2453.5	0.2965	8.666	8.370
21	2.4882	54.483	88.10	2539.3	2451.2	0.3107	8.644	8.333
22	2.6453	51.418	92.28	2541.1	2448.8	0.3249	8.622	8.297
23	2.8111	48.548	96.46	2542.9	2446.4	0.3391	8.600	8.261
24	2.9858	45.858	100.65	2544.7	2444.0	0.3532	8.578	8.225
25	3.1699	43.337	104.83	2546.5	2441.7	0.3672	8.557	8.189
26	3.3639	40.973	109.01	2548.3	2439.3	0.3812	8.535	8.154
27	3.5681	38.754	113.19	2550.1	2436.9	0.3952	8.514	8.119
28	3.7831	36.672	117.37	2551.9	2434.6	0.4091	8.493	8.084
29	4.0092	34.716	121.55	2553.7	2432.2	0.4229	8.473	8.050
30	4.2470	32.878	125.73	2555.5	2429.8	0.4368	8.452	8.015
32	4.7596	29.526	134.09	2559.2	2425.1	0.4642	8.411	7.947
34	5.3251	26.560	142.45	2562.8	2420.3	0.4916	8.371	7.880
36	5.9479	23.929	150.81	2566.3	2415.5	0.5187	8.332	7.814
38	6.6328	21.593	159.17	2569.9	2410.8	0.5456	8.294	7.748
40	7.3849	19.515	167.53	2573.5	2406.0	0.5724	8.256	7.683
42	8.2096	17.664	175.89	2577.1	2401.2	0.5990	8.218	7.619
44	9.1124	16.011	184.25	2580.6	2396.4	0.6255	8.182	7.556
46	10.0990	14.534	192.62	2584.2	2391.6	0.6517	8.145	7.494
48	11.1770	13.212	200.98	2587.8	2386.8	0.6779	8.110	7.432
50	12.352	12.027	209.34	2591.3	2381.9	0.7038	8.075	7.371
52	13.631	10.963	217.71	2594.8	2377.1	0.7296	8.040	7.311
54	15.022	10.006	226.07	2598.3	2372.3	0.7553	8.007	7.251
56	16.533	9.1448	234.44	2601.8	2367.4	0.7808	7.973	7.193
58	18.171	8.3683	242.81	2605.3	2362.5	0.8061	7.940	7.134
60	19.946	7.6672	251.18	2608.8	2357.7	0.8313	7.908	7.077
65	25.042	6.1935	272.12	2617.5	2345.4	0.8937	7.830	6.936
70	31.201	5.0395	293.07	2626.1	2333.0	0.9551	7.754	6.799
75	38.595	4.1289	314.03	2634.6	2320.6	1.0158	7.681	6.665

T °C	P kPa	v_g $\mathrm{m^3\,kg^{-1}}$	h_f $\mathrm{kJ\,kg^{-1}}$	h_g $\mathrm{kJ\,kg^{-1}}$	h_{fg} $\mathrm{kJ\,kg^{-1}}$	s_f $\mathrm{kJ\,kg^{-1}\,K^{-1}}$	s_g $\mathrm{kJ\,kg^{-1}\,K^{-1}}$	s_{fg} $\mathrm{kJ\,kg^{-1}\,K^{-1}}$
80	47.414	3.4052	335.01	2643.0	2308.0	1.0756	7.611	6.536
85	57.867	2.8258	356.01	2651.3	2295.3	1.1346	7.543	6.409
90	70.182	2.3591	377.04	2659.5	2282.5	1.1929	7.478	6.285
95	84.608	1.9806	398.09	2667.6	2269.5	1.2504	7.415	6.165
100	101.325	1.6732	419.06	2675.5	2256.5	1.3069	7.354	6.047
105	120.9	1.4184	440.3	2683.4	2243.1	1.363	7.295	5.932
110	143.38	1.2093	461.4	2691.1	2229.6	1.419	7.238	5.819
115	169.18	1.0358	482.6	2698.6	2216.0	1.474	7.183	5.709
120	198.67	0.89121	503.8	2705.9	2202.1	1.528	7.129	5.601
125	232.24	0.77003	525.1	2713.1	2188.0	1.582	7.077	5.496
130	270.28	0.66800	546.4	2720.1	2173.7	1.635	7.026	5.392
135	313.23	0.58173	567.7	2726.9	2159.1	1.687	6.977	5.290
140	361.54	0.50845	589.2	2733.4	2144.3	1.739	6.929	5.190
145	415.68	0.44596	610.6	2739.8	2129.2	1.791	6.883	5.092
150	476.16	0.39245	632.2	2745.9	2113.7	1.842	6.837	4.995
155	543.5	0.34646	653.8	2751.8	2098.0	1.892	6.793	4.900
160	618.23	0.30678	675.5	2757.4	2082.0	1.943	6.749	4.807
165	700.93	0.27243	697.2	2762.8	2065.6	1.992	6.707	4.714
170	792.19	0.24259	719.1	2767.9	2048.8	2.042	6.665	4.623
175	892.6	0.21658	741.0	2772.7	2031.7	2.091	6.624	4.534
180	1002.8	0.19384	763.1	2777.2	2014.2	2.139	6.584	4.445
185	1123.5	0.17390	785.2	2781.4	1996.2	2.188	6.545	4.357
190	1255.2	0.15636	807.4	2785.3	1977.9	2.236	6.506	4.270
195	1398.8	0.14089	829.8	2788.8	1959.0	2.283	6.468	4.185
200	1554.9	0.12721	852.3	2792.0	1939.7	2.331	6.430	4.100
210	1907.7	0.10429	897.6	2797.3	1899.6	2.425	6.356	3.932
220	2319.6	0.086092	943.6	2800.9	1857.4	2.518	6.284	3.766
230	2797.1	0.071503	990.2	2802.9	1812.7	2.610	6.213	3.603
240	3346.9	0.059705	1037.6	2803.0	1765.4	2.702	6.142	3.440
250	3976.2	0.050083	1085.8	2800.9	1715.2	2.794	6.072	3.279
260	4692.3	0.042173	1135.0	2796.6	1661.6	2.885	6.002	3.117
270	5503	0.035621	1185.3	2789.7	1604.4	2.977	5.930	2.954
280	6416.6	0.030153	1236.9	2779.9	1543.0	3.069	5.858	2.789
290	7441.8	0.025555	1290.0	2766.7	1476.7	3.161	5.783	2.622
300	8587.9	0.021660	1345.0	2749.6	1404.6	3.255	5.706	2.451
310	9865.1	0.018335	1402.2	2727.9	1325.7	3.351	5.624	2.273
320	11284	0.015471	1462.2	2700.6	1238.4	3.449	5.537	2.088
330	12858	0.012979	1525.9	2666.0	1140.2	3.552	5.442	1.890
340	14601	0.010781	1594.5	2621.8	1027.3	3.660	5.336	1.676
350	16529	0.008802	1670.9	2563.6	892.7	3.778	5.211	1.433
360	18666	0.006949	1761.7	2481.5	719.8	3.917	5.054	1.137
370	21044	0.004954	1890.7	2334.5	443.8	4.111	4.801	0.690
373.95	22064	0.003106	2084.3	2084.3	0.0	4.407	4.407	0.000

C.2 Sorted by Pressure

P kPa	T °C	v_g $m^3\,kg^{-1}$	h_f $kJ\,kg^{-1}$	h_g $kJ\,kg^{-1}$	h_{fg} $kJ\,kg^{-1}$	s_f $kJ\,kg^{-1}\,K^{-1}$	s_g $kJ\,kg^{-1}\,K^{-1}$	s_{fg} $kJ\,kg^{-1}\,K^{-1}$
0.6117	0.01	205.991	0.00	2500.9	2500.9	0.0000	9.156	9.156
0.80	3.8	159.640	15.81	2507.8	2492.0	0.0575	9.057	8.999
1.00	7.0	129.178	29.30	2513.7	2484.4	0.1059	8.975	8.869
1.50	13.0	88.321	54.56	2524.7	2470.1	0.1951	8.828	8.633
2.00	17.5	66.987	73.43	2532.9	2459.4	0.2606	8.723	8.462
2.50	21.1	54.240	88.42	2539.4	2451.0	0.3118	8.642	8.330
3.00	24.1	45.653	100.98	2544.8	2443.8	0.3543	8.576	8.222
3.50	26.7	39.466	111.82	2549.5	2437.7	0.3906	8.521	8.130
4	29.0	34.791	121.39	2553.7	2432.3	0.4224	8.473	8.051
5	32.9	28.185	137.75	2560.7	2423.0	0.4762	8.394	7.918
6	36.2	23.733	151.48	2566.6	2415.2	0.5208	8.329	7.808
7	39.0	20.524	163.35	2571.7	2408.4	0.5590	8.275	7.715
8	41.5	18.099	173.84	2576.2	2402.4	0.5925	8.227	7.635
9	43.8	16.199	183.25	2580.2	2397.0	0.6223	8.186	7.564
10	45.8	14.670	191.81	2583.9	2392.1	0.6492	8.149	7.500
12	49.4	12.358	206.91	2590.3	2383.4	0.6963	8.085	7.389
14	52.5	10.691	219.99	2595.8	2375.8	0.7366	8.031	7.295
16	55.3	9.4306	231.57	2600.6	2369.1	0.7720	7.985	7.213
18	57.8	8.4431	241.96	2605.0	2363.0	0.8036	7.944	7.140
20	60.1	7.6480	251.42	2608.9	2357.5	0.8320	7.907	7.075
24	64.1	6.4453	268.15	2615.9	2347.7	0.8819	7.844	6.962
28	67.5	5.5778	282.66	2621.8	2339.2	0.9247	7.791	6.866
32	70.6	4.9215	295.52	2627.1	2331.6	0.9623	7.745	6.783
36	73.3	4.4072	307.09	2631.8	2324.7	0.9958	7.705	6.709
40	75.9	3.9930	317.62	2636.1	2318.4	1.0261	7.669	6.643
45	78.7	3.5759	329.62	2640.9	2311.2	1.0603	7.629	6.569
50	81.3	3.2400	340.54	2645.2	2304.7	1.0912	7.593	6.502
60	85.9	2.7317	359.91	2652.9	2292.9	1.1454	7.531	6.386
70	89.9	2.3648	376.75	2659.4	2282.7	1.1921	7.479	6.287
80	93.5	2.0871	391.71	2665.2	2273.5	1.2330	7.434	6.201
90	96.7	1.8694	405.20	2670.3	2265.1	1.2696	7.394	6.125
100	99.6	1.6939	417.50	2674.9	2257.4	1.3028	7.359	6.056
101.325	100.0	1.6732	419.06	2675.5	2256.5	1.3069	7.354	6.047
110	102.3	1.5495	428.8	2679.2	2250.3	1.3330	7.327	5.994
120	104.8	1.4284	439.4	2683.1	2243.7	1.3609	7.298	5.937
130	107.1	1.3253	449.2	2686.6	2237.5	1.3868	7.271	5.884
140	109.3	1.2366	458.4	2690.0	2231.6	1.4110	7.246	5.835
150	111.3	1.1593	467.1	2693.1	2226.0	1.4337	7.223	5.789
160	113.3	1.0914	475.4	2696.0	2220.7	1.4551	7.201	5.746
170	115.1	1.0312	483.2	2698.8	2215.6	1.4753	7.181	5.706
180	116.9	0.97747	490.7	2701.4	2210.7	1.4945	7.162	5.668
190	118.6	0.92924	497.9	2703.9	2206.0	1.5127	7.144	5.631
200	120.2	0.88568	504.7	2706.2	2201.5	1.5302	7.127	5.597
220	123.3	0.81007	517.6	2710.6	2193.0	1.5628	7.095	5.532
240	126.1	0.74668	529.6	2714.6	2185.0	1.5930	7.066	5.473
260	128.7	0.69273	540.9	2718.3	2177.4	1.6210	7.039	5.418
280	131.2	0.64624	551.4	2721.7	2170.3	1.647	7.015	5.368

P kPa	T °C	v_g $m^3\,kg^{-1}$	h_f $kJ\,kg^{-1}$	h_g $kJ\,kg^{-1}$	h_{fg} $kJ\,kg^{-1}$	s_f $kJ\,kg^{-1}\,K^{-1}$	s_g $kJ\,kg^{-1}\,K^{-1}$	s_{fg} $kJ\,kg^{-1}\,K^{-1}$
300	133.5	0.60576	561.4	2724.9	2163.5	1.672	6.992	5.320
320	135.7	0.57017	570.9	2727.8	2157.0	1.695	6.970	5.275
340	137.8	0.53864	579.9	2730.6	2150.7	1.717	6.950	5.233
360	139.8	0.51050	588.5	2733.2	2144.7	1.738	6.931	5.193
380	141.8	0.48522	596.8	2735.7	2139.0	1.758	6.913	5.155
400	143.6	0.46238	604.7	2738.1	2133.4	1.777	6.896	5.119
420	145.4	0.44165	612.3	2740.3	2128.0	1.795	6.879	5.085
440	147.1	0.42274	619.6	2742.4	2122.8	1.812	6.864	5.052
460	148.7	0.40542	626.6	2744.4	2117.7	1.829	6.849	5.020
480	150.3	0.38950	633.5	2746.3	2112.8	1.845	6.834	4.990
500	151.8	0.37481	640.1	2748.1	2108.0	1.860	6.821	4.960
600	158.8	0.31558	670.4	2756.1	2085.8	1.931	6.759	4.828
700	164.9	0.27277	697.0	2762.8	2065.8	1.992	6.707	4.715
800	170.4	0.24034	720.9	2768.3	2047.4	2.046	6.662	4.616
900	175.4	0.21489	742.6	2773.0	2030.5	2.094	6.621	4.527
1000	179.9	0.19436	762.5	2777.1	2014.6	2.138	6.585	4.447
1200	188.0	0.16326	798.3	2783.7	1985.4	2.216	6.522	4.306
1400	195.0	0.14078	830.0	2788.8	1958.9	2.284	6.468	4.184
1600	201.4	0.12374	858.5	2792.8	1934.4	2.344	6.420	4.077
1800	207.1	0.11037	884.5	2795.9	1911.4	2.398	6.378	3.980
2000	212.4	0.099585	908.5	2798.3	1889.8	2.447	6.339	3.892
2200	217.2	0.090698	930.9	2800.1	1869.2	2.492	6.304	3.812
2400	221.8	0.083244	951.9	2801.4	1849.6	2.534	6.271	3.737
2600	226.0	0.076899	971.7	2802.3	1830.7	2.574	6.241	3.667
2800	230.1	0.071429	990.5	2802.9	1812.4	2.611	6.212	3.602
3000	233.9	0.066664	1008.3	2803.2	1794.8	2.646	6.186	3.540
3500	242.6	0.057058	1049.8	2802.6	1752.8	2.725	6.124	3.399
4000	250.4	0.049776	1087.5	2800.8	1713.3	2.797	6.070	3.273
4500	257.4	0.044059	1122.2	2797.9	1675.7	2.862	6.020	3.158
5000	263.9	0.039446	1154.6	2794.2	1639.6	2.921	5.974	3.053
6000	275.6	0.032448	1213.9	2784.6	1570.7	3.028	5.890	2.862
7000	285.8	0.027378	1267.7	2772.6	1505.0	3.122	5.815	2.692
8000	295.0	0.023526	1317.3	2758.7	1441.4	3.208	5.745	2.537
9000	303.3	0.020490	1363.9	2742.9	1379.1	3.287	5.679	2.392
10000	311.0	0.018030	1408.1	2725.5	1317.4	3.361	5.616	2.255
11000	318.1	0.015990	1450.4	2706.3	1255.9	3.430	5.555	2.124
12000	324.7	0.014264	1491.5	2685.4	1194.0	3.497	5.494	1.997
13000	330.9	0.012780	1531.5	2662.7	1131.2	3.561	5.434	1.873
14000	336.7	0.011485	1571.0	2637.9	1066.9	3.623	5.373	1.750
15000	342.2	0.010338	1610.2	2610.7	1000.5	3.685	5.311	1.626
16000	347.4	0.009309	1649.7	2580.8	931.1	3.746	5.246	1.501
17000	352.3	0.008371	1690.0	2547.5	857.5	3.808	5.179	1.371
18000	357.0	0.007502	1732.1	2509.8	777.7	3.872	5.106	1.234
19000	361.5	0.006677	1777.2	2466.0	688.9	3.940	5.026	1.086
20000	365.7	0.005865	1827.2	2412.3	585.1	4.016	4.931	0.916
21000	369.8	0.004996	1887.6	2338.6	451.0	4.106	4.808	0.702
22064	373.95	0.003106	2084.3	2084.3	0.0	4.407	4.407	0.000